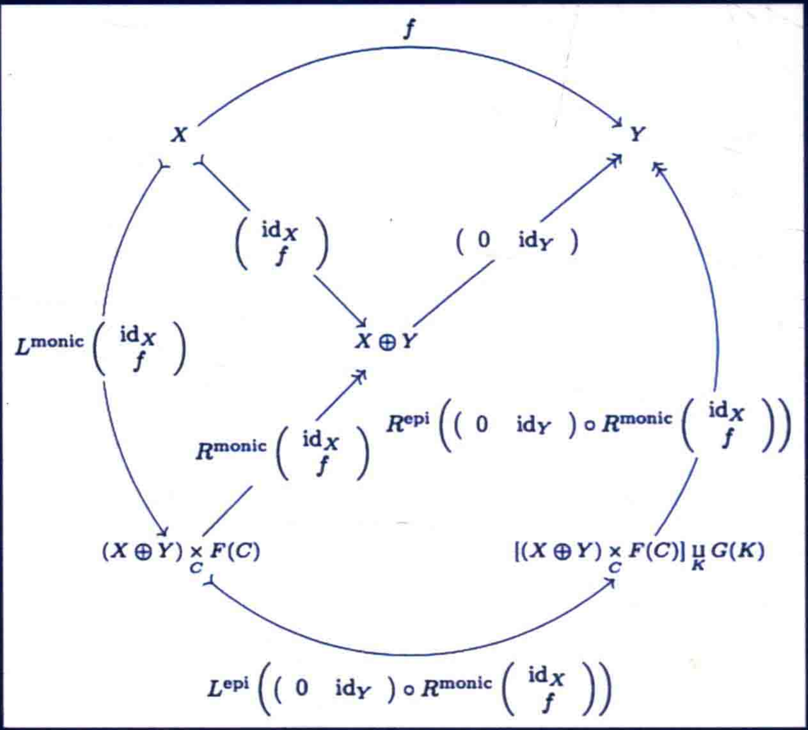
MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Introduction to Abelian Model Structures and Gorenstein Homological Dimensions



Marco A. Pérez

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To the memory of my grandmother,
María Romelia.

Preface

The goal of this work is to obtain new model structures in homological algebra. The idea is to construct a pair of compatible complete cotorsion pairs

$$(\mathcal{A} \cap \mathcal{W}, \mathcal{B}) \quad \text{and} \quad (\mathcal{A}, \mathcal{B} \cap \mathcal{W})$$

related to a specific homological dimension, and then apply *Hovey Correspondence* to obtain an Abelian model structure where the classes of cofibrant, fibrant, and trivial objects coincide with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$, respectively.

The contributions presented in this book can be split into two parts. In the first one, we study the projective, injective, and flat dimensions of objects in the category $\mathsf{Ch}(R)$ of chain complexes of modules, in order to obtain compatible and complete cotorsion pairs for each of these dimensions, via the application of techniques such as the *zig-zag argument*. We recall several model structures on the category of chain complexes, obtained from the classes of projective, injective and flat modules, such as the *flat* and the *degreewise projective model structures* mentioned below, and then we will present their corresponding generalizations to homological dimensions.

In the second part, we restrict our attention to the case where R is a Gorenstein ring, in which another type of homological algebra appears in $\mathsf{Mod}(R)$ and $\mathsf{Ch}(R)$, described by the notions of Gorenstein-projective, Gorenstein-injective and Gorenstein-flat dimensions. M. Hovey and J. Gillespie constructed model structures on $\mathsf{Mod}(R)$ having the classes of Gorenstein-projective, Gorenstein-injective and Gorenstein-flat modules among the cofibrant and fibrant objects. We will see how to generalize Hovey's arguments to get new Abelian model structures on $\mathsf{Mod}(R)$ and $\mathsf{Ch}(R)$ from Gorenstein-homological dimensions.

Some history, from Salce to Hovey

Nowadays, probably among the most important objects in the realm of homological algebra are the cotorsion pairs. First introduced by Luigi Salce in the category of groups, they were rediscovered by Edgar E. Enochs for the category of modules. Colloquially, two classes of modules form a cotorsion pair if they are orthogonal to each other with respect to the first extension functor

$\mathsf{Ext}^1_R(-,-)$. This definition, which seems to be very simple at first sight, turns out to have very deep applications in several branches of mathematics, the Representation Theory of Algebras being probably the most favored.

Two of the most important episodes representing the impact of cotorsion pairs are linked to some homological conjectures. For instance, the flat cover conjecture explicitly first stated in 1981 by Enochs in his paper *Injective and flat covers, envelopes and resolvents*, remained unsolved for about 20 years. It asserts the existence of a flat cover for every module. This was proven to be true in 2001 by Enochs, thanks to some contributions by Paul Eklof and Jan Trlifaj, and simultaneously and independently by L. Bican and R. El Bashir.

The theory of cotorsion pairs was also used by Lidia Angeleri-Hügel and Octavio Mendoza in *[AHM09]* to establish a validity criterion for the second finitistic dimension conjecture, which states that the little finitistic dimension of every finite Artin algebra is finite. This has been proved to be true in some particular cases, such as for finite dimensional monomial algebras. The proof of the cited criterion uses the fact, proved by S. T. Aldrich, E. E. Enochs, O. M. G. Jenda and L. Oyonarte, that the class of modules with projective dimension at most n (with n some positive integer) is the left half of a complete cotorsion pair.

In 2002, Mark Hovey established a correspondence between the theories of cotorsion pairs and model structures. Namely, Hovey proved that given an Abelian model structure on a bicomplete Abelian category, it is possible to construct two complete cotorsion pairs from the classes of cofibrant, fibrant and trivial objects of the given model. Moreover, the converse is also true, that is, if we are given three classes of objects in such a category forming two compatible and complete cotorsion pairs, then it is possible to obtain a unique Abelian model structure such that the classes of cofibrant, fibrant and trivial objects coincide with the given classes.

Hovey's results provide an easy method to construct model structures on categories such as modules or chain complexes. Concerning complexes over a ring or a ringed space, James Gillespie introduced the notions of differential graded chain complexes with respect to a class of modules. One important example of an Abelian model structure obtained by *Hovey Correspondence* is given by Gillespie in the paper *The flat model structure on* $\mathrm{Ch}(R)$, published in 2004, by proving that the classes of flat and dg-flat chain complexes are the left halves of two compatible complete cotorsion pairs.

Another interesting example of a model structure on chain complexes obtained by *Hovey Correspondence* was given by D. Bravo, E. E. Enochs, A. Iacob, O. M. G. Jenda and J. Rada in their article *Cotorsion pairs in C(RMod)*, published in 2012. There they proved that the classes of degreewise and exact degreewise projective chain complexes are the left halves of two compatible complete cotorsion pairs.

In this book, we will continue the path started by the mentioned authors, constructing along the way new model structures from classical and Gorenstein homological dimensions, and also generalizing and/or reproving some important existing results. Let us be more specific about this in the following paragraphs.

Outline

In *Part 1* we introduce some categorical preliminaries and notations. One of the purposes of this work is to present most of the definitions and results in a categorical setting, so it is necessary to recall the definitions and notations of the universal constructions most used in Category Theory. We present each construction along with a diagram so it will be easier to understand and recall the concept for the reader who is not familiar with categories. In *Chapter 2* we give a review of Abelian and Grothendieck categories. We present some notions known in Relative Homological Algebra, such as left and right resolutions with respect to a class of objects, covers and envelopes, and left and right homological dimensions. This material is studied in detail in *Chapter 3*, where we also present extensions functors in two ways, namely, via cohomologies or using *Baer description*. We also show how complexes with bounded projective or injective dimension can be expressed as exact complexes whose cycles have projective or injective dimension with the same bound. Torsion functors, on the other hand, are left for *Chapter 4*. This chapter begins with the definition of (closed and symmetric) monoidal categories, and then continues with a detailed exposition of several examples. We also study two tensor products of chain complexes, and describe the flat complexes corresponding to each tensor.

In *Part 2*, *Chapter 7*, we present the investigation done by M. Hovey that connects the theories of cotorsion pairs and model categories. We begin by giving the definition of weak factorization systems, as the core notion of that of a model structure. Roughly speaking, a weak factorization system is given by two classes of morphisms in a category $\mathcal{C}$ such that they have a lifting property with respect to each other and satisfy a certain factorization axiom. A morphism $f\colon X \to Y$ lifts with respect to a morphism $g\colon W \to Z$ in a commutative square

$$\begin{array}{ccc} X & \longrightarrow & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & Z \end{array}$$

if there exists a morphism $d\colon Y \to W$ such that the resulting inner triangles commute:

$$\begin{array}{ccc} X & \longrightarrow & W \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle d} & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & Z \end{array}$$

On the other hand, the equality $\mathrm{Ext}^1_{\mathcal{C}}(X,Y) = 0$ means that every short exact sequence of the form

$$0 \to Y \xrightarrow{\alpha} Z \xrightarrow{\beta} X \to 0$$

splits, that is there is a morphism $\beta' \colon X \to Z$ such that $\beta \circ \beta' = \mathrm{id}_X$. In other words, we have a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & Z \\ \downarrow & \nearrow{\scriptstyle \beta'} & \downarrow{\scriptstyle \beta} \\ X & = & X \end{array}$$

meaning that $0 \to X$ lifts with respect to β. Hovey noticed this particular behavior, and established a correspondence for constructing a certain type of model structure from a pair of cotorsion pairs satisfying a compatibility condition. Since this correspondence is of vital importance in this work, we think it is pertinent to present a proof, although in a particular way via the concept of Abelian factorization systems.

Concerning cotorsion pairs, in *Chapter 6* we present a proof of Eklof and Trlifaj Theorem in Grothedieck categories, originally proven in the category of modules. We later present some methods developed by J. Gillespie to induce certain cotorsion pairs in chain complexes from a cotorsion pair in an Abelian category $\mathcal{C}$. These induced cotorsion pairs involve classes of complexes which are basically relative versions of differential graded projective and differential graded injective complexes.

Part 3 is devoted to the study of the relationship between model structures and classical homological dimensions. We construct six model structures on the category of chain complexes, namely: the n-projective, n-injective, n-flat, degreewise n-projective, degreewise n-injective, and degreewise n-flat model structures. We start with the projective dimension in the category of chain complexes. In *Chapter 9* we work with the category $\mathsf{Mod}(\mathfrak{R})$ of modules over a ringoid $\mathfrak{R}$. We prove that the class $\mathsf{Proj}_n(\mathfrak{R})$ of n-projective modules over $\mathfrak{R}$ (*i.e.*, with projective dimension $\leqslant n$) is the left half of a complete and hereditary cotorsion pair. We will apply this result to deduce that there is a unique model structure on the category $\mathsf{Ch}(R)$ of chain complexes over R where the trivial objects are given by the exact complexes, and the trivially cofibrant objects by the n-projective complexes. We also study properties of the homotopy category of this model structure, and from them we deduce other ways to compute extension functors of modules. Later in the following sections, we present and construct the degreewise n-projective model structure mentioned before. The case $n = 0$ proved by J. Rada and coauthors is based on a famous theorem by I. Kaplansky on projective modules. They prove that there exists a unique Abelian on $\mathsf{Ch}(R)$ where the trivial objects are given by the exact chain complexes, and the cofibrant objects by the complexes of projective modules. We generalize this result to projective dimensions $n > 0$.

Using properties of injective objects in Grothendieck categories and the theory of induced cotorsion pairs, in *Chapter 8* we obtain the dual of the two previous model structures, namely, the n-injective and degreewise n-injective model structures on complexes over Grothendieck categories.

Chapter 10 is devoted to constructing the n-flat and degreewise n-flat model structures on $\mathsf{Ch}(R)$. We apply the contents from *Chapter 4*, and generalize some zig-zag procedures in chain complexes, to show that the class of complexes with flat dimension $\leqslant n$ is the left half of a complete cotorsion pair. This class will be the class of trivially cofibrant objects of an Abelian model structure on $\mathsf{Ch}(R)$. We also prove a similar result for the class of complexes of modules with flat dimension $\leqslant n$.

In *Part 4*, we focus our attention on a special type of Grothendieck categories known as Gorenstein categories. They represent, in some sense, a generalization of the homological algebra occuring for modules over a Gorenstein ring. On such categories we can construct two model structures with the classes of Gorenstein-projective and Gorenstein-injective objects as the cofibrant and fibrant objects, respectively. These classes are also examples of Frobenius categories, and we will see how to apply this information to present the homotopy categories of the previous model structures as a generalization of the stable module category of a quasi-Frobenius ring. We also study Gorenstein-injective dimensions in Gorenstein categories, and obtain for each $n \geqslant 0$ a model structure having the objects with Gorenstein-injective dimension $\leqslant n$ as the class of fibrant objects. All of this theory is developed in *Chapters 11* and *12*.

The Gorenstein-projective dimension and related model structures need to be studied in the more particular setting provided by modules and complexes over a Gorenstein ring. In *Chapter 13* we obtain model structures on both categories having the class of objects with Gorenstein-projective dimension $\leqslant n$ as the class of cofibrant objects.

A study of Gorenstein-flat modules is presented in *Chapter 14* on modules and complexes over a Gorenstein ring R. We construct a unique Abelian model structure on R such that the cofibrant objects are the modules with Gorenstein-flat dimension $\leqslant n$, and where the trivial objects are given by the class of modules with finite flat dimension. We show how to obtain the analog of this structure for chain complexes, but in order to fulfill that goal, we will define and study properties of an alternative notion of cotorsion pairs of complexes.

Intended readers

This book is based in the author's Ph.D. thesis, and so it contains some new results in relative homological algebra and model category theory. This could make us think the contents are very technical. However, it has been written as self-contained as possible, and the chapters are organized to help the reader to learn the necessary stuff step by step. This work is intended for graduated students entering the field, and secondly, it can serve as a very detailed reference for researchers. Some known results by other authors also also re-proven, using different arguments or from a pedagogical point of view. Moreover, some folklore results, which would not be easy to find in the literature, are also proven. Most of the proofs are presented in detail, and all the necessary references are given.

The author's results were stated and proven between 2011 and 2013, so the materials presented in this book are a little bit behind the current state of the field. That is why some *Further Reading* sections are included at the end of some chapters, to recommend more references for a wider understanding of the subject.

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Acknowledgements

This book is mostly based on my Ph.D. thesis, so I would like to start by thanking my doctoral advisor Prof. André Joyal, for all his help and constant support throughout my studies. This work has grown out of a better understanding of the relation between cotorsion pairs and abelian model structures, and so it has been an excellent place to apply the knowledge I acquired during my Masters and Ph.D. studies. Our meetings to discuss these topics have been of invaluable help. Also, I am grateful to my thesis evaluators: Edgar E. Enochs (Ed), Mark Hovey, and Franco Saliola, who made useful observations and corrections on the first draft.

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