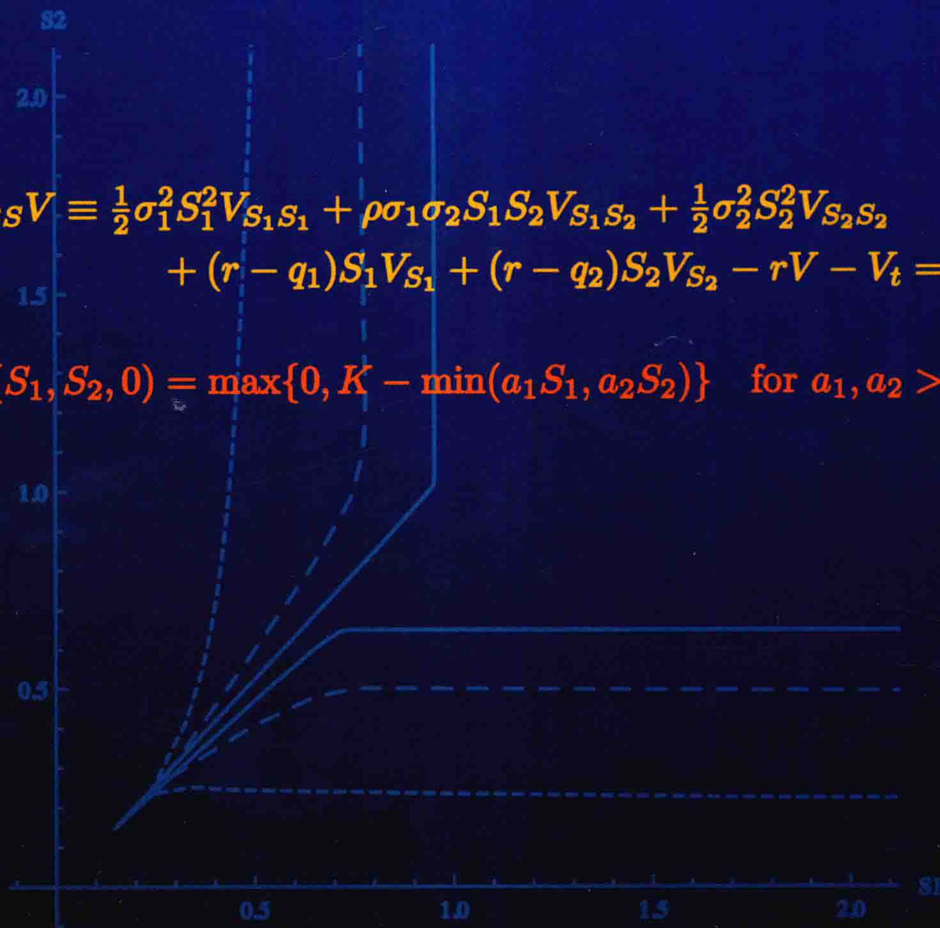
THE TIME-DISCRETE METHOD OF LINES FOR OPTIONS AND BONDS

A PDE Approach



$$\mathcal{L}_{BS}V \equiv \tfrac{1}{2}\sigma_1^2 S_1^2 V_{S_1S_1} + \rho\sigma_1\sigma_2 S_1 S_2 V_{S_1S_2} + \tfrac{1}{2}\sigma_2^2 S_2^2 V_{S_2S_2} + (r-q_1)S_1V_{S_1} + (r-q_2)S_2V_{S_2} - rV - V_t = 0$$

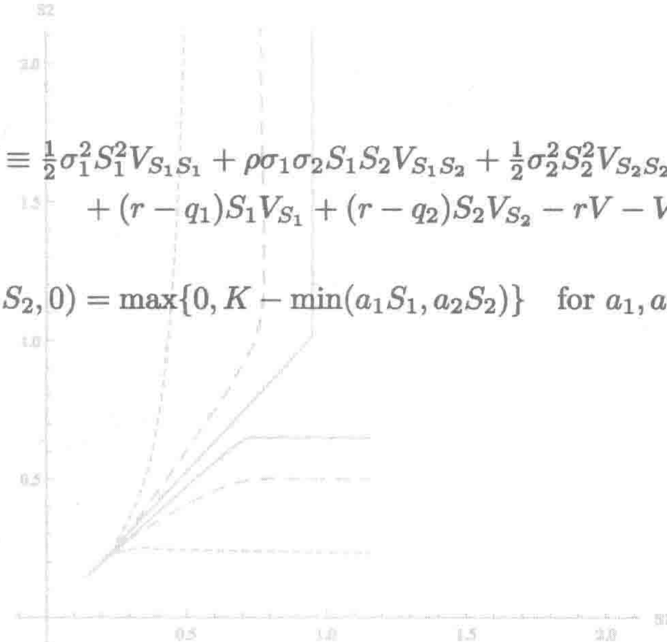
$$V(S_1, S_2, 0) = \max\{0, K - \min(a_1S_1, a_2S_2)\} \quad \text{for } a_1, a_2 > 0$$

Gunter H. Meyer

World Scientific

THE TIME-DISCRETE METHOD OF LINES FOR OPTIONS AND BONDS

A PDE Approach



$$\mathcal{L}_{BS}V \equiv \frac{1}{2}\sigma_1^2 S_1^2 V_{S_1S_1} + \rho\sigma_1\sigma_2 S_1 S_2 V_{S_1S_2} + \frac{1}{2}\sigma_2^2 S_2^2 V_{S_2S_2} + (r-q_1)S_1V_{S_1} + (r-q_2)S_2V_{S_2} - rV - V_t = 0$$

$$V(S_1, S_2, 0) = \max\{0, K - \min(a_1S_1, a_2S_2)\} \quad \text{for } a_1, a_2 > 0$$

Gunter H. Meyer

Georgia Institute of Technology, USA

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

THE TIME-DISCRETE METHOD OF LINES FOR OPTIONS AND BONDS
A PDE Approach

Copyright © 2015 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

ISBN 978-981-4619-67-7

In-house Editor: Qi Xiao

Printed in Singapore

THE TIME-DISCRETE METHOD OF LINES FOR OPTIONS AND BONDS

A PDE Approach

Preface

In these notes we discuss some of the issues which arise when the partial differential equations (pdes) modeling option and bond prices are to be solved numerically. A great variety of numerical methods for this task can be found in textbooks and the research literature, and all are effective for pricing Black Scholes options on a single asset and bonds based on a one-factor interest rate model, particularly when prices far enough away from expiration are to be found.

However, there are financial applications where pde methods have to cope with uncertainty in the problem description, with rapidly changing solutions and their derivatives, with nonlinearities, non-local effects, vanishing diffusion in the presence of strong convection, and the "curse of dimensionality" due to multiple assets and factors in the financial model. All these complications are inherent in the pde formulation and must be overcome by whatever numerical method is chosen to price options and bonds accurately and efficiently.

The focus of these notes is on identifying and discussing these complications, to remove uncertainty in the pde model due to incomplete or inconsistent boundary data, and to illustrate through extensive simulations the computational problems which the pde model presents for its numerical solution. We concentrate on pricing models which have been presented in the literature, and for which results have been obtained with various numerical methods for specific applications. Here we shall search out problem settings where the complications in the pde formulation can be expected to degrade the numerical results.

We do not discuss different numerical methods for the pdes of finance and their effectiveness in solving practical problems. All simulations will be carried out with the time-discrete method of lines. We view it as a flexible

此为试读,需要完整PDF请访问: www.ertongbook.com

tool for solving low-dimensional time dependent pricing problems in finance. It is based on a solution method which for simple American puts and calls is algorithmically equivalent to the Brennan-Schwartz method. For multi-dimensional problems it is combined with a locally one-dimensional line Gauss Seidel iteration. The method is introduced in detail in these notes. We think that it performs well enough that we offer the results of our simulations as benchmark data for a variety of challenging financial applications to which competing numerical methods could be applied.

The notes are intended for readers already engaged in, or contemplating, solving numerically partial differential equations for options and bonds. The notes are written by a mathematician, but not for mathematicians. They are "applied" and intended to be accessible for graduates of programs in quantitative and computational finance and practicing quants who have learned about numerical methods for the Black Scholes equation, the bond equation, and their generalisations. But we also assume that the readers have not had a particular exposure to, or interest in, the theory of partial differential equations and the mathematical analysis of numerical method for solving them. There are many sources in the textbook and research literature on both aspects, a notable example being the rigorous textbook/monograph of Achdou and Pironneau [1]. But such sources would appeal more to specialists than to the readership we hope to reach.

These notes are not a text for a course on numerical methods for the pdes of finance, nor are they intended to answer real questions in finance. The pde models discussed at length below are drawn from various published sources and readers are referred to the cited literature for their derivation and discussion. On occasion the models will be modified because finance suggests it or mathematics demands it. They will be solved numerically with assumed data, frequently chosen to accentuate the severity of the application and the behavior of the solution. Financial implications of our results will mostly be ignored. Although not a textbook, this book could serve as a reference for an advanced applied course on pdes in finance because it discusses a number of topics germain to all numerical methods in this field regardless of whether the method of lines is ever mentioned.

Both the pde problem specification and its numerical solution will be of interest. We shall assume that given a financial model for the evolution of an asset price, a volatility, an interest rate, etc., the pricing pde can be derived under specific assumptions reflecting or approximating the market reality. The validity of the pde, usually a time dependent diffusion equation, is not considered in doubt.

However, the pde does not constitute the whole model. The pde is only solvable if the problem for it is (in the language of mathematics) "well posed", meaning that it has a solution, that the solution is unique, and that the solution varies continuously with the data of the problem. If the pde is to be solved numerically, then in general it must be restricted to a finite computational domain. In order to be well posed an initial condition and the behavior on the boundary of the computational domain must be given. The initial condition is usually the pay-off of the option or the value of the bond at expiration, both of which are unambiguous and consistent with the pricing problem being well posed. For options the pay-off tends to introduce singularities into the solution or its derivatives which can make pricing of even simple options like puts and calls near maturity a challenging numerical problem.

In contrast to the certainty about initial conditions, the proper choice of boundary conditions can be complicated. The structure of the pdes arising in finance can exert a dominant influence on what boundary conditions can be given, and where, to retain a well-posed problem. This is often not a question of finance but relates to a fairly recent and still incomplete mathematical analysis of admissible boundary conditions for so-called degenerate evolution equations. While the mathematical theory is likely to be too abstract for the intended readership of these notes, we hope that sufficient operational information has been extracted from it to give guidance for choosing admissible boundary conditions. The most difficult case arises when for lack of better information a modification of the pde itself is used to set a boundary condition on a computational boundary. This aspect of the pde model is independent of the numerical method chosen for its solution. However, mathematically admissible boundary conditions are usually not unique. Some preserve the structure of the boundary value problem required for the intended numerical method but are inconsistent financially, while other admissible boundary conditions may be harder to incorporate into a numerical method but may yield solutions which are less driven by where we place the computational boundary. Simulation seems the only choice to check how uncertain boundary data will affect the solution.

The book has seven chapters. Section 1.1 of the first chapter reflects the view that once a well-posed mathematical model is accepted, then the solution is unambiguously determined and its mathematical properties must be acceptable on financial grounds. The examples of this section are based on elementary mathematical manipulations of the Black Scholes equation and its extensions and formally prove results which often are obvious from arbitrage arguments. Section 1.2 concentrates on a discussion of admissible boundary conditions for degenerate pricing equations in finance. It introduces the Fichera function as a tool to determine where on the boundary of its domain of definition the pricing equation has to hold, and where unrelated conditions can be imposed. We illustrate the application of the Fichera function for a number of option problems including cases where boundary conditions at infinity have to be set. We then consider the problem of conditions on the boundary of a finite computational domain where financial arguments often do not provide boundary conditions. We show that reduced versions of the pde can provide acceptable tangential boundary conditions known as Venttsel boundary conditions.

Chapter 2 introduces the method of lines for a scalar diffusion equation with one or two free boundaries. It will then be combined with a line Gauss-Seidel iteration to yield a locally one-dimensional front tracking method for time-discretized multi-dimensional diffusion problems subject to fixed and free boundary conditions.

Chapter 3 discusses in detail the numerical solution of the one- dimensional problems with the so-called Riccati transformation. It is closely related to the Thomas algorithm for the tri-diagonal matrix equation approximating linear second order two-point boundary value problems and is equally efficient.

The next four chapters consist of numerical simulations of options and bonds. The numerical method chosen for the simulations is always the method of lines of the preceding two chapters, but the numerical method intrudes little on the discussion of the pde model and the quality of its solutions.

Chapters 4 and 5 deal with European and American options priced with the Black Scholes equation. Comparisons with analytic solutions, where available, give the sense that such options can be computed to a high degree of accuracy even near expiration.

Chapter 6 concentrates on fixed income problems based on general one-factor interest rate models, including those admitting negative interest rates.

The experience gained with scalar diffusion problems is brought to bear in Chapter 7 on options for two assets, including American max and min options. It is shown that on occasion front tracking algorithms for American options can benefit by working in polar coordinates when the early exercise boundary on discrete rays is a well defined function of the polar angle.

The last example of an American call with stochastic volatility and interest rate suggests that the application of a locally one-dimensional front tracking method remains feasible in principle but presents hardware and programming challenges not easily met by the linear Fortran programs and the desktop computer used for our simulations.

Throughout these notes we give, besides graphs, a lot of tabulated data obtained with the method of lines for a variety of financial problems. Such data may prove useful as benchmark results for the implementation of the method of lines or other numerical methods for related problems. As already stated, the financial parameters are only assumed, but our numerical simulations appear to be robust over large parameter ranges for all the models discussed here. This may help when the method of lines is applied as a general forward solver in a model calibration.

Finally, we will admit that the choice of financial models treated here is more a reflection on past exposure, experience and taste than an orderly progression from simple to complicated models, or from elementary to relevant models. Our judgment of what questions are relevant in finance is informed by the texts of Hull [38] and Wilmott [64], while the more mathematical thoughts were inspired by the texts of Kwok [46] and Zhu, Wu and Chern [67], which we value for their breadth and mathematical precision. So far, the method of lines has proved to be a flexible and effective numerical method for pricing options and bonds, and as demonstrated in a concurrent monograph of Chiarella et al. [17], it can hold its own against some competing numerical methods for pdes in finance. MOL cannot work for all problems, but we do not hide its failures.

G. H. Meyer

Acknowledgment

I am grateful to the School of Mathematics of the Georgia Institute of Technology for remaining my scientific home in the years since my retirement. Interaction with colleagues, teaching the occasional course, and having access to the resources of the School and the Institute have made this time an unbroken, and unexpected, sabbatical.

The most important resource has been the cooperation of Ms. Annette Rohrs of the School in producing this book. In spite of a varied and steadily expanding workload, she has once again managed to turn scribbled and ever changing notes into a camera-ready book. Without her help I would not have started this project and could not have finished it. Thank you again, Annette.

Contents

此为试读,需要完整PDF请访问: www.ertongbook.com

Chapter 1

Comments on the Pricing Equations in Finance

The two dominant pricing equations of these notes are the Black Scholes equation for the price $V(S,t)$ of an option

$$\mathcal{L}_{BS}V \equiv \frac{1}{2}\sigma^2 S^2 V_{SS} + (r-q)SV_S - rV - V_t = 0 \tag{1.1}$$

for $0 < S < \infty$ and $t \in (0,T]$, and the bond equation for the price $B(r,t)$ of a discount bond

$$\mathcal{L}_B B \equiv \frac{1}{2}w^2 B_{rr} + (u - \lambda w)B_r - rB - B_t = 0 \tag{1.2}$$

for, usually, $0 < r < \infty$ and $t \in (0,T]$, where $t = T - \tau$ for calendar time τ denotes the time to expiry. Both equations are augmented by the values $V(S,0)$ and $B(r,0)$ at expiration $t = 0$ and by boundary conditions on V and B which are determined by the specific application. The aim is to find "a solution" of the pricing equation which also satisfies the given side conditions.

These equations, and their multi-factor generalizations, are special forms of a so-called evolution equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{m} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{m} b_i(x,t)u_{x_i} + c(x,t)u + d(x,t)u_t = f(x,t) \tag{1.3}$$

where $A = (a_{ij}(x,t))$ is a symmetric matrix, $d(x,t) \neq 0$, and where $(x,t) = (x_1,\dots,x_m,t)$ denotes the $m+1$ independent variables. Typically x belongs to an open bounded or unbounded set $D(t)$ in R_m with boundary $\partial D(t)$, and t belongs to an interval $(0,T]$. We remark that time dependent domains $D(t)$ are common in the free boundary formulation of American options.

If the matrix A in (1.3) is positive definite then the equation is known as a parabolic or, somewhat imprecisely, a diffusion equation. In finance the matrix is often non-negative definite which complicates its analysis. For simplicity we shall call (1.3) a diffusion equation even if A is only semi-definite.

The most famous example of a diffusion equation is the simple heat equation for conductive heat transfer

$$\mathcal{L}u \equiv u_{xx} - u_t = 0$$

which often can be solved analytically. It is well known that the Black Scholes equation (1.1) for constant parameters can be transformed to the heat equation through a change of variable, and that the corresponding Green's function solutions are the Black Scholes formulas for various European options (see [33] and Example 1.1). When analytic solutions are not available one usually resorts to numerical solutions.

The partial differential equations of finance are mathematical models which are derived in many textbooks under specific market assumptions and simplifications which do not necessarily reflect the market reality (see, e.g. [64]). But once the model equations and their initial and boundary conditions are accepted, one also has to accept the qualitative and quantitative behavior of their solutions which is entirely determined by the structure of the mathematical problem and not the application. One cannot assume a priori that the mathematical solution will show all the properties which are obvious for financial reasons. Instead, one has to prove that the mathematical solutions are consistent with financial arguments. If not, the model would have to be changed.

1.1 Solutions and their properties

The user of the partial differential equations of finance tends to assume that the mathematical problem has a solution. The user also tends to have a strong intuitive sense of whether an approximate or numerical solution is "correct". To a mathematician the problem is not quite so simple because the meaning of solution is ambiguous. A problem may not have a solution in one sense but may well have a unique solution if the class of admissible functions is broadened by allowing certain types of discontinuities. Moreover, an approximate solution may well solve a closely related problem while the actual formulation does not have any solution.

There is a comprehensive mathematical theory on the existence, uniqueness and properties of solutions of parabolic problems (see, e.g. [28], [47],

[48], and research on its extensions continues unabated. Here, to characterize solutions of differential equations without going into technical detail, the following few (very loose) definitions are convenient:

Definition. A function u is smooth if it has as many continuous derivatives as are needed for the operations to which it is subjected.

Definition. A classical solution of (1.3) subject to an initial condition at $t = 0$ and to boundary conditions on $\partial D(t)$ is a function which is continuous on the closed set $\overline{D(t)} \times [0, T]$, smooth on $D(t) \times (0, T]$, and which satisfies point for point the equation and the initial and boundary conditions.

Definition. A weak solution is a function which satisfies the equation (1.3) and the side conditions in an "integral sense" (see, e.g. Section 1.2.1).

For example, the Black Scholes formula for a European put is a classical solution of the Black Scholes equation (1.1) and the pay-off and boundary conditions

$$V(S,0) = \max\{0, K - S\}$$
$$V(0,t) = Ke^{-rt}, \quad \lim_{S\to\infty} V(S,t) = 0$$

while the Black Scholes formula for a European digital call with initial condition

$$V(S,0) = \begin{cases} 0 & S < K \\ 1 & S \geq K \end{cases}$$

is not a classical solution because $V(S,0)$ is discontinuous at the strike price K. Similarly, the solution for an up (or down) and out barrier option generally has a discontinuity at expiration at the barrier and is therefore only a weak solution. We mention that classical solutions are always weak solutions.

Many of the conceptual problems due to discontinuous initial/boundary conditions can be circumvented if we think of approximating the data by continuous functions. For example, the digital call $V(S,t)$ may be defined as

$$V(S,t) = \lim_{\epsilon\to 0} V_\epsilon(S,t)$$

where

$$V_\epsilon(S,0) = \begin{cases} 0 & S < K - \epsilon \\ (S - (K - \epsilon))/(2\epsilon) & K - \epsilon \leq S \leq K + \epsilon. \\ 1 & S > K + \epsilon \end{cases}$$

此为试读,需要完整PDF请访问: www.ertongbook.com