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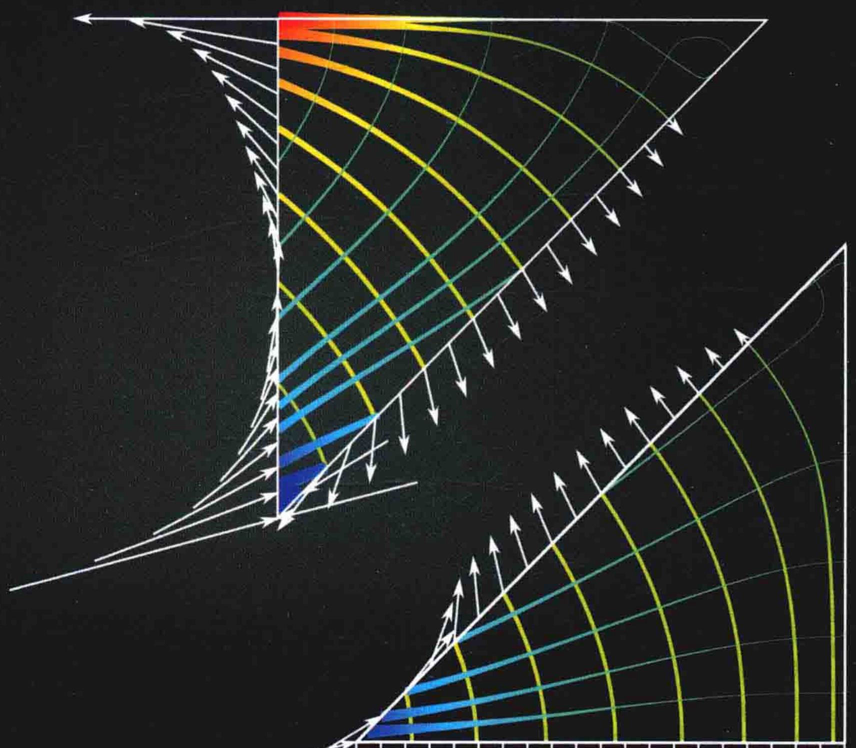
J.P. MOITINHO DE ALMEIDA AND EDWARD A. W. MAUNDER



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Preface

This book is the result of many years of joint work and fun with equilibrium finite element formulations. Although they are often regarded as the ugly duckling of computational mechanics, we know that they have characteristics that are particularly attractive. It thus became our mission to spread the word that a 'strongly equilibrated finite element' is not a contradiction.

It all started a long time ago, in the late 1960s, when Edward, then a PhD student in civil engineering at Imperial College, attended a lecture by Fraeijs de Veubeke.[1] In retrospect he now has some questions that might have been interesting at the time, but he also admits that the recognition of the practical relevance of the equilibrium formulations for continua that were presented only really came later, while doing structural design of reinforced concrete using stress fields obtained from displacement based finite elements.

Edward was also fortunate to have John Henderson as supervisor and later as a friend, colleague and father figure. John was a polymath with a firm belief in the benefits of a proper mathematical foundation to structural analysis, from vector spaces to algebraic topology. He encouraged the transfer of knowledge gained from the analysis of aircraft structures to the analysis of civil engineering structures, using concepts of static-kinematic duality in the pursuit of equilibrium via flexibility methods.

Zé Paulo's discovery of how to impose strong forms equilibrium, and as consequence the discovery of Fraeijs de Veubeke's and John Henderson's work, happened in 1985, also at Imperial College. His objective was to figure out the characteristics of the solutions of simple elastoplastic models which could enforce either equilibrium or compatibility, leading to interesting complementarities in the results.

Bruce Iron's book *Techniques of Finite Elements*, published in 1980, with its clear, imaginative and friendly style, focused on 'enabling the understanding of mathematical and physical concepts, because effective trouble-shooting is best achieved with such harmony', also had a very strong influence on both authors.

The stars were thus aligned for equilibrium when we first met in the office of David Lloyd Smith at Imperial College, sometime in 1992.[2] After many papers, projects and conferences, where stress fields and spurious kinematic modes were dissected to exhaustion, we decided to collect our ideas in a book.

1 Probably in the Department of Aeronautics, where he also attended memorable lectures given by Kelsey, of Argyris and Kelsey fame!
2 Following the publication of 'An alternative approach to the formulation of hybrid equilibrium finite elements'.

The resulting text provides a comprehensive presentation of an equilibrium formulation of the finite element method, principally with application to 2D, 3D and plate flexure problems in structural mechanics, when strains can be assumed to be infinitesimally small. Equal weight is given to the construction of stress fields that strongly satisfy equilibrium within and between elements, as well as displacements on element boundaries that are 'broken' or discontinuous at vertices of 2D plate elements or edges of solid elements.

We present up-to-date developments which enable dual analyses of models to be undertaken, either as a means of verification or as an alternative source of output that may be more directly useful to design engineers.

The book starts with a simple introduction of the concepts involved, followed by a historical introduction of equilibrium in the context of finite element analysis, and comparison with other formulations. We discuss the details of the equilibrium formulation in the context of modelling linear elastic static and dynamic behaviour with particular emphasis on the associated problems of spurious kinematic modes. A more mathematical justification of the formulation is included, where we propose a relevant functional to be used in the variational analysis of the saddle point problem, and attempt to explain its significance in engineering terms to a non-mathematician. We then proceed to present methods to recover complementary conforming and equilibrating solutions from each other, and show how the dual nature of such solutions enables bounds to be enumerated on global or local quantities of interest. The text concludes by opening routes to extending the formulation in order to simulate various forms of non-linear behaviour.

We make particular effort to explain the more mathematical concepts in straightforward terms which we hope will be understandable by the intended readership, namely senior undergraduates of engineering and applied mathematics, graduate researchers and practising engineers with an interest in verification, duality and safe structural design.

The topics are illustrated with a range of numerical examples which have been carefully designed to be simple, but of just sufficient complexity to highlight particular features.

The book contains two appendices: the first to summarize the fundamental equations of structural mechanics, and the second to serve as a companion to the computer programs that were developed in the course of writing the book. These programs are available on request to the first author and we intend to publish them, when they are more mature, under an open source licence.

The time from initial thoughts, in 2009, to the formal proposal, submitted in April 2011 and accepted in March 2012, was almost as long as that to complete the text. As usual, longer than we had anticipated.

We thank, first and foremost, our families for their patience and moral support. Our gratitude also extends to all the colleagues and friends who, directly or indirectly, and in many different ways, have helped us in developing our ideas before or during the writing of this book: in particular to Orlando Pereira and Pierre Beckers who read the manuscript, as well as to Angus Ramsay, Antonio Huerta, Bassam Izzuddin, Bill Harvey, Carlos Tiago, David Lloyd Smith, Eduardo Arantes e Oliveira, João Teixeira de Freitas, John Robinson, Luiz Fernando Martha, Pedro Diez, Philippe Bouillard and Pierre Ladevèze.

We do not forget to thank all the people in the editorial and production teams at Wiley, for their patience and advice throughout the preparation of this book. A special mention should be made regarding the helpful suggestions by the copy editor, Chris Cartwright.

Our recognition obviously includes all those who we have forgotten – sorry for that.

We hope you enjoy this book as much as we enjoyed writing it.

Lisbon and Exeter, March 2016
Zé Paulo and Edward

List of Symbols

The general meaning of the most commonly used symbols is given in this list. Other meanings may be specified in the text.

$\boldsymbol{A}$	Nullspace of $\boldsymbol{D}^T$
$\mathcal{A}$	Area of a face
$\boldsymbol{b}$ or b_α	(Generalized) body force vector or component
$\boldsymbol{C}$	Equilibrium/compatibility matrix for the mixed formulations
$\boldsymbol{\chi}$ or $\chi_{\alpha\beta}$	Curvature vector or component
$\mathcal{D}$ or $\mathcal{D}^\star$	Differential compatibility or equilibrium operator
$\boldsymbol{D}$	Equilibrium/compatibility matrix for the hybrid formulations
$d_\square$	Degree of a given polynomial approximation
$\boldsymbol{\Delta}$	Rigid body displacement
E	Young's modulus
$\boldsymbol{\varepsilon}$ or $\varepsilon_{\alpha\beta}$	(Generalized) strain vector or component
ϵ	Bound of the error of a pair of solutions
$\mathcal{F}$	Element flexibility matrix
$\boldsymbol{f}$	Material flexibility matrix
Γ	Boundary of the problem
$\boldsymbol{\gamma}$ or γ_α	Shear strain vector or component
$\boldsymbol{K}$	Element stiffness matrix
$\boldsymbol{k}$	Material stiffness matrix
k_α^β	Signature function for an edge ($\beta = +$ or $-$) or a face ($\beta = 1 \ldots 3$)
L_α	Area coordinate on a face
$\lambda(\xi)$	Lagrange multiplier function
l	Length of a side
$Lk(V)$	Link of vertex V
$\mathcal{L}()$	Local output
$\boldsymbol{m}$ or $m_{\alpha\beta}$	Distributed bending moment vector or component
$\mathcal{M}$	Mobility matrix
$\mathcal{N}$	Boundary normal operator
$n_\square$	Number of parameters of a given approximation
$N(V)$	Simplicial neighbourhood of vertex V
$\lVert\square\rVert$	Energy norm (may be defined for a displacement, strain, or a stress field)
ν	Poisson's ratio

Ω	Domain of the problem
ω	Eigenfrequency
φ	Internal angle at a vertex of a triangle or dihedral angle at an edge of a tetrahedron
$\bar{\varphi}$	$\cot \varphi$
Π	Total potential energy
Π_c	Total complementary energy
Π_c^G	Generalized complementary energy
Ψ	Partition of unity function
$\boldsymbol{\psi}$ or ψ^α	Interpolation matrix or function
$\boldsymbol{q}$ or q_α	Distributed shear force vector or component
ρ	Mass density
$\boldsymbol{S}$	Stress approximation matrix
$\boldsymbol{\sigma}$ or $\sigma_{\alpha\beta}$	(Generalized) stress vector or component
$\hat{\boldsymbol{s}}$	Stress approximation parameters
$\mathcal{S}$	Set (including vector space) of statically admissible stress fields
T	Kinetic energy density
$\boldsymbol{t}$ or t_α	(Generalized) boundary traction vector or component
$\boldsymbol{\theta}$ or θ_α	Rotation vector or component
$\mathcal{T}$	Set (including vector space) of side tractions
U	Strain energy
U_c	Complementary strain energy
$\boldsymbol{u}$ or u_α	(Generalized) displacement vector or component
$\mathcal{U}_k$	Set (including vector space) of kinematically admissible displacement fields
$\boldsymbol{v}$	Boundary displacement
$\boldsymbol{V}$	Boundary displacement approximation matrix
$\hat{\boldsymbol{v}}$	Boundary displacement approximation parameters
$\mathcal{V}$	Set (including vector space) of boundary displacements
V	Work done by the applied forces
V_c	Work done by the imposed displacements
W	Strain energy density
W_c	Complementary strain energy density
w	Transverse displacement of a plate
ξ	Coordinate on the boundary of an element
$\square_e$ or $\square_m$	Vector associated with element e or boundary entity m
$\square_{e_\alpha}$ or $\square_{m_\alpha}$	Component of vector associated with element e or boundary entity m

Contents

1

Introduction

1.1 Prerequisites

A very concise description of this book is that it

> presents a methodology to predict and explain the distribution of forces and deflections that develop within a loaded structure.

For a layman who is unfamiliar with structural analysis, this description requires further explanations of many important points, namely: What is a structure? What are the forces within the structure? What are loads? Why and how do they get distributed?

We will not try to address these questions. For their answers a basic book on structural analysis, for example, Coates *et al.* (1988); Hibbeler (2008) or Marti (2013) will provide the necessary knowledge on the concepts used to describe structural behaviour – equilibrium, compatibility and constitutive relations – as well as the variables involved – forces, displacements, stresses and strains.

For all but the simplest problems, the mathematical equations used to describe the relations between these structural variables cannot be solved in a closed form. Of the various techniques that are used to obtain approximate solutions of these equations we will focus our attention on the application of a particular technique, the finite element method (FEM). Though it is possible to gain an understanding of FEM concepts solely from the information that will be presented in this text, it is more convenient to start with a basic book on finite element procedures, for example, Fish and Belytschko (2007).

We will therefore assume that the reader has a basic knowledge of the problems of structural analysis, namely of the fundamental equations of solid mechanics and, at least, some understanding of the procedures involved in the application of the FEM, most probably using a conventional displacement based formulation.

Such a reader, probably with an engineering background in the context of aeronautical, civil or mechanical engineering applications, given a title which includes 'equilibrium'

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and 'finite elements' might rather wonder: '*Why another book? The finite element method is well known and it provides solutions that satisfy equilibrium. Doesn't it?*' The fact is that in most cases it doesn't, since only an approximate form of equilibrium is achieved by displacement based finite element formulations.

Our text presents a way to obtain solutions that are different from the 'usual' ones, because they exactly satisfy equilibrium. Nevertheless, since they normally omit the strict enforcement of compatibility conditions, it is not possible to say *a priori* which will be better. They just fail in different ways.

We believe that exploiting the complementarity of the two approaches allows for an interpretation of the results that is more profound than what is possible with a single type of analysis, naturally providing the tools for the assessment of their quality.

That, in the end, is our goal. Explaining in detail how equilibrated solutions can be obtained is just a step towards it.

1.2 What Is Meant by Equilibrium? Weak to Strong Forms

We expect the reader to understand what is meant by a free body, and being in a state of equilibrium, that is, the forces and their moments sum to zero. However, although checks on equilibrium at the global or overall level of a structure, for example, as represented by its finite element model, are commonly undertaken, deeper investigations into local levels of equilibrium become more problematic.

In FEM there are various shades of meaning, and perhaps expectation, when considering local equilibrium. The concept of a free body normally starts at the level of an infinitesimal element in a *continuum* (i.e. strong equilibrium between body forces and stresses), which is itself a mathematical abstraction – since we ignore the microscopic structure of the material.

Then the concept moves to the level of a single finite element, and then it may move back to another mathematical abstraction – a *node* of an element, where we invoke the concept of *nodal forces* (i.e. corresponding to a weak form of equilibrium between statically equivalent forces).

It is relevant here to note that the concept of a nodal force may not be explicitly mentioned in texts on finite elements, and we are aware of commercially available software where nodal forces are not available to the user, but only stress contours and tables of stresses at particular points!

In practice some confusion exists, and engineers may be unaware of the 'subtle' distinctions between these different levels of equilibrium of free bodies, and their significance to the analysis of a finite element model. We frequently hear of engineers who look blank when advised that local equilibrium is usually violated – they appear to have a firm conviction that equilibrium is being satisfied in all necessary aspects. Their first response might be: '*Does it matter if there are local violations?*'

An appropriate reply might be: 'It all depends on your needs and how well you know the distribution of the loads.' This is a matter of judgement, but we would advise that engineers, when faced with many uncertainties, can proceed with more confidence knowing that their analysis provides complete equilibrium. Local violations can be regarded as residual loads that are equilibrated by the errors in stress, and such loads are made orthogonal to the displacements allowed by a conforming model. By refining the model, the solutions converge, even when residual loads persist.

Our starting point is the fact that conventional finite element analyses 'provide solutions that equilibrate the equivalent nodal forces', where the adjective *equivalent* plays a central role that is often disregarded in the more basic introductions to the FE method.

Effectively there is equilibrium of equivalent nodal forces in the solutions provided by most FE programs. We will discuss in detail what that means and we will conclude that, in most cases, there are no nodal forces as such. Energetically consistent nodal forces are defined, which are required to produce the same work as the real forces and stresses for all displacements considered. But, in general, this is not sufficient to guarantee equilibrium in a strong, or pointwise, sense.

This happens because only a finite subset of the possible displacements can be included in a given model and the solution space is generally infinite, therefore equilibrium is imposed on an average, or weak form. Generally

the solutions provided by displacement based FE models do not enforce the equilibrium conditions at every point of the domain and/or its boundary.

Our objective is to present in this book a methodology whose models produce solutions that strongly verify all equilibrium conditions. As always there is a drawback for every new approach. In this case the gain in terms of equilibrium will imply a loss in terms of compatibility, which will only be imposed in a weak form.

We will not pretend that these equilibrium formulations are always better than their displacement based counterparts, as each formulation locally enforces one set of conditions, while imposing a weak form of the other.

1.3 What Do We Gain From Strong Forms of Equilibrium?

The complementary nature of these formulations is, in our opinion, the strongest reason for considering solutions obtained from *both approaches*. It does not matter which one is considered first, as the different approximate solutions that they produce are complementary, in the sense that they satisfy complementary equations in a strong and in a weak form.

As we will show, this complementarity can be used in a natural way to assess the quality of the solutions, and to drive a mesh adaptation process, deciding where it is important to have more, or fewer elements. From a practical point of view it is also relevant to point out that equilibrium solutions have the advantage of being immediately usable as a safe basis for design of ductile structures, when the Static Theorem of Limit Analysis can be invoked (fib, 2013; Marti, 2013; Nielsen and Hoang, 2010).

In particular, equilibrium solutions give us a more rational way of accounting for stress concentrations, especially when they arise due to mathematical singularities where the structural geometry has been simplified, for example, at re-entrant corners.