DIFFERENTIAL FORMS IN MATHEMATICAL PHYSICS

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Preface

The radical change of methods used in mathematical physics (quantum field theory and elementary particle physics, solid state physics, theory of dynamical systems etc.) influenced by the great power of modern mathematics calls for a monograph such as this which is aimed primarily at making available to physical scientists the mathematical machinery related to differentiable manifold ideas relevant to physics. The reader will find that the "geometry spirit" working methods provide a new marvellously unified set of tools for an alternate description of natural phenomena which goes beyond the description obtained in terms of analytical methods.

The concept of a differential form is associated with the differentiable manifold idea – one of the keystones of contemporary mathematics – appearing in the theory of partial differential equations, algebraic topology and differential geometry. In studying geometry or physical laws by tensor methods it turns out that tensor analysis is not adequate. It demands a nonsingular coordinate system with respect to which one can give the components of vectors and tensors. However, according to the definition of a differentiable manifold, a single nonsingular coordinate system is not enough to cover a manifold. Therefore, in a general differentiable manifold it will be impossible to describe a tensor field by giving its components with respect to a single set of coordinates. Consequently the components of a tensor field are not as important as the concept that can be abstracted from them: the intrinsic representation of a tensor. Of all the types of tensor fields, the skew-symmetric covariant ones are intrinsically represented by differential forms. Physical theories, in particular Maxwell's theory, the Yang–Mills theory, the theory of relativity, but also physical laws of thermodynamics and analytical mechanics (symplectic mechanics) can be given a neat and concise formulation in terms of them.

Differential forms have also been used by de Rham to express a deep relation between the topological structure of a manifold and certain aspects of vector analysis on a manifold. Elie Cartan used differential forms to develop his approach to differential systems and Riemannian geometry. Our main concern in this book is therefore with a straightforward exposition of vector analysis on manifolds which is designed as a comprehensive elementary approach to Cartan's and de Rham's work. In particular we are also going to exhibit that the generalization of Stokes' theorem and the divergence theorem to general manifolds is very clumsy unless one employs a systematic development of the calculus of differential forms. In taking up this calculus and its use in formulating integration theory we provide physicists with contemporary mathematical methods. At the same time, we wish to contribute towards a conceptually intelligible and transparent vector-analytical description of physical laws. Therefore, many physical applications have been interspersed in the presentation throughout this book. In particular, the last two chapters are completely devoted to physical sciences.

During the past years (1969–77) the author has delivered lectures on differential forms and their applications in mathematical physics in various universities to audiences consisting of mathematicians, mathematical physicists, theoretical physicists, mathematically inclined experimental physicists and engineers. This book constitutes an extended and improved version of the material presented in these lectures. Nevertheless, it should be emphasized that one of the goals of the present book is to develop an intuition and working knowledge of the subject "differential forms in mathematical physics" without insisting on an extremely high degree of mathematical rigour or precision as would be required if the audience consisted of mathematicians alone.

The prerequisites required for a "reasonable" reading of this volume include a working knowledge of set theory, linear algebra, calculus as well as of undergraduate physics education.

I am deeply indebted to Claire for her constant encouragement and her valuable criticism. It is also my pleasant duty to thank my colleagues Professors Holmann and Rummler (Fribourg University), Mislin and Jost (Swiss Institute of Technology), Crumeyrolle (Université Paul-Sabatier, Toulouse), Kalina and Lawrynowicz (Instytut matematyczny, Polskiej Akademii Nauk, Łodź), E. Heil (Technische Hochschule, Darmstadt) and Geissler (Hamburg University) for their critical reading and many helpful suggestions. My special thanks to Ernst Seligmann, who devoted considerable time and effort in providing me, in a very cooperative way, with substantial scientific material relevant to this book.

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PART I

BASIC CONCEPTS

CHAPTER 1

TOPOLOGICAL PRELIMINARIES

Summary. Topological spaces are the objects of study of this chapter. A topological space consists of a set E and a "topology" on E which may be defined equivalently in terms of open sets, closed sets or neighbourhoods. Open set topologies or closed set topologies, although devoid of the intuitive appeal of the neighbourhood topology, are logically simpler and therefore provide a better method of defining a topology (Section 2).

Many properties of topological spaces depend on the distribution of the open sets in the space. If a topological space has "few" open sets, it is more likely to be first or second countable. On the other hand, if a topological space has "many" or "enough" open sets, sequences of this space have unique limits, provided that a suitable "separation property" is postulated for this space. This is achieved, if the axiomatization of a topological space is supplemented by a separation axiom. We restrict ourselves, throughout this book to spaces which are defined by the Hausdorff separation axiom (Section 5).

Formally topology is characterized as the study of those properties of spaces (as for instance compactness or connectedness, cf. Section 5) which are not changed under homeomorphisms (Section 4), that is, topology is the study of topological invariants.

Prerequisites. Set theory (set operations, cartesian products, mappings).

1. Introduction

The notion of topology gives sense to the intuitive ideas of nearness and continuity. It appears that there are equivalent ways of defining a topology: In terms of *open sets*, or of *closed sets* or using as primitive notion the notion of *neighbourhood of a point*. The former definitions, although devoid of the intuitive appeal of the neighbourhood definition, are logically simpler and therefore provide a better method of defining a topology.

We summarize, omitting most proofs, in Chapter 1 the preliminary topological material necessary for this book and refer the reader for a more detailed account to the quoted reference books.

2. Topological spaces

A topological space is a non-empty set E together with a family $\mathfrak{T} = (U_i \mid i \in I)$ of subsets of E satisfying the following axioms:

(01) $E \in \mathfrak{T}$, $\emptyset \in \mathfrak{T}$ *(where $\emptyset$ denotes the empty set).*

(02) *The intersection of any finite number of sets in $\mathfrak{T}$ belongs to $\mathfrak{T}$, i.e.*

$$J \text{ finite}, J \subset I \Rightarrow \bigcap_{i \in J} U_i \in \mathfrak{T}.$$

(03) *The union of any number of sets in $\mathfrak{T}$ belongs to $\mathfrak{T}$*, i.e.

$$J \subset I \Rightarrow \bigcup_{i \in J} U_i \in \mathfrak{T}.$$

The elements of $\mathfrak{T}$ are called *$\mathfrak{T}$-open sets*, or simply open sets in E. The pair $(E, \mathfrak{T})$ is called a topological space.

Example 2.1. The class $\mathfrak{T} = \{E, \emptyset\}$, consisting of E and $\emptyset$ alone is itself a topology called the *indiscrete topology*. $(E, \mathfrak{T})$ is then called an indiscrete topological space.

Example 2.2. Let $\mathfrak{T} = \mathfrak{P}(E)$ denote the family of all subsets of E. Observe that $\mathfrak{P}(E)$ satisfies the axioms (01)–(03) for a topology on E. This topology is called the *discrete topology*; the pair $(E, \mathfrak{T})$ is called a discrete topological space.

Example 2.3. Let $E = \mathbf{R}$ be the real line. A *topology on* $\mathbf{R}$ can be defined as follows: For any $x \in \mathbf{R}$, consider the open intervals (a, b) containing x, that is the class

$$\mathfrak{T} = \{U_i = (a_i, b_i) \mid a_i, b_i \in \mathbf{R}\}.$$

If $a_i = b_i$, then $(a_i, b_i) = \emptyset$.

By a straightforward verification $\mathfrak{T}$ is seen to satisfy the axioms (01)–(03). This topology is referred as the *usual topology* on $\mathbf{R}$. Similarly, the usual topology on $\mathbf{R}^n$, the product set of n copies of the set $\mathbf{R}$, is given by the family $\mathfrak{T}$ of all open sets $U_1 \times U_2 \times \cdots \times U_n$, where $U_i = (a_i, b_i)$, $1 \leq i \leq n$, are open intervals in $\mathbf{R}$. We shall always assume the usual topology on $\mathbf{R}$ and $\mathbf{R}^n$ unless otherwise specified.

Let $(E, \mathfrak{T})$ be a topological space. A subset A of E is *closed* if its complement $C_E A := \{x \in E \mid x \notin A\}$ is an open set.

From the axioms (01), (02) and (03) of a topological space and De Morgan's laws one infers: The family $\bar{\mathfrak{T}} = (A_i \mid i \in I)$ of closed subsets of E satisfies the following conditions:

(C1) E and $\emptyset$ are closed sets, i.e. $E \in \bar{\mathfrak{T}}$, $\emptyset \in \bar{\mathfrak{T}}$.

(C2) The union of any finite number of sets in $\bar{\mathfrak{T}}$ belongs to $\bar{\mathfrak{T}}$:

$$J = \{i_1, i_2, \ldots, i_n\} \subset I \quad \Rightarrow \quad \bigcup_{i=i_1}^{i_n} A_i \in \bar{\mathfrak{T}}.$$

(C3) The intersection of any number of sets in $\bar{\mathfrak{T}}$ belongs to $\bar{\mathfrak{T}}$, i.e.

$$J \subset I \quad \Rightarrow \bigcap_{i \in J} A_i \in \bar{\mathfrak{T}}.$$

From (01), (02) and (03) we infer that, by duality, an equivalent definition of a topological space in terms of closed sets is possible. We denote this topological space by $(E, \bar{\mathfrak{T}})$.

Let $x \in E$ be a point in a topological space E. Any subset V of E containing an open set U such that $x \in U$ is called a *neighbourhood of* x denoted by $V = V(x)$. In particular, any open set U is a neighbourhood of each of its points. The class of all neighbourhoods of $x \in E$, denoted by $\mathfrak{V}(x)$, is called the *fundamental neighbourhood system* of x.

Example 2.4. Let $x \in \mathbf{R}$. Then each closed interval $[x - \delta, x + \delta]$ with centre x, is a neighbourhood of x since it contains the open interval $(x - \delta, x + \delta)$ containing x.

The following properties of neighbourhoods may be used to define a topology on E:

(V1) $\mathfrak{V}(x)$ is not empty and x belongs to each member of $\mathfrak{V}$, i.e.

$$x \in E, \quad V \in \mathfrak{V}(x) \quad \Rightarrow \quad x \in V.$$

(V2) $(\forall V_1, V_2 \in \mathfrak{V}(x))(\exists V_3 \in \mathfrak{V}(x))\colon (V_3 \subset V_1 \cap V_2)$.

(V3) If $V \in \mathfrak{V}(x)$, $y \in V$, there is a $U \in \mathfrak{V}(y)$ such that $U \subset V$.

It is seen that the structures of open set topology, closed set topology and neighborhood topology determine one another: so topology may be developed using either as a starting point. By virtue of these different axiomatizations of topological spaces, the word "topology" will be used to denote these equivalent structures. Thus a topological space carries all these structures, and may be defined by one of them. Finally, let $(E, \mathfrak{T})$ be a topological space. A class $\mathfrak{B}$ of open subsets of E, $\mathfrak{B} \subset \mathfrak{T}$, is a *base* for the topology $\mathfrak{T}$ iff: