lecture notes in pure and applied mathematics

exceptional lie algebras

N. Jacobson

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about the book . . .

This volume presents a set of models for the exceptional Lie algebras over algebraically closed fields of characteristic 0 and over the field of real numbers. The models given are based on the algebras of Cayley numbers (octonions) and on exceptional Jordan algebras. They are also valid for characteristics $p \neq 2$. The book also provides an introduction to the problem of forms of exceptional simple Lie algebras, especially the exceptional D_4's, E_6's, and E_7's. These are studied by means of concrete realizations of the automorphism groups.

Exceptional Lie Algebras is a useful tool for the mathematical public in general—especially those interested in the classification of Lie algebras or groups—and for theoretical physicists.

about the author . . .

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Preface

The main purpose of these notes is to provide a set of models for the exceptional Lie algebras over algebraically closed fields of characteristic 0 and over the field $\mathbb{R}$ of real numbers. The models we give are based on the algebras of Cayley numbers (octonions) and on exceptional Jordan algebras. These are valid also for arbitrary fields of characteristic $\neq 2, 3$. Our second aim is to give an introduction to the problem of forms of exceptional simple Lie algebras, especially, the exceptional D_4's, E_6's and E_7's. These are studied by means of concrete realizations of the automorphism groups. We omit a discussion of the earlier results on G_2's and F_4's due respectively to the author and to Tomber. Further developments along these lines are due to the author (for D_4's), to Allen (for D_4's), to Ferrar (for E_6's and E_7's). The papers containing these results are given in the Bibliography.

The first half of these notes (§1-7) were written in 1958. §8 was written in 1965 and most of this has appeared in Ferrar's paper [1]. The remainder of the notes were written in 1970. Section 11 on the Killing forms is the result of joint work with Ferrar.

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Introduction

We shall be interested in certain Lie algebras, particularly, certain exceptional simple Lie algebras defined by Jordan algebras and by alternative algebras. We recall that an algebra is Jordan if the base field is of characteristic not two and the multiplication $a \cdot b$ satisfies

(1) $\qquad a \cdot b = b \cdot a \ , \qquad (a^2 \cdot b) \cdot a = a^2 \cdot (b \cdot a) .$

In such an algebra we write R_a for the mapping $x \longrightarrow x \cdot a$. Then the second identity states that $[R_a R_{a^2}] = 0$ where $[AB]$ is the Lie product $AB - BA$ in the associative algebra of linear transformations. The axioms (1) have a number of consequences which will be recalled when needed. Here we note

(2) $\qquad [[R_a R_b] R_c] = R_{A(b,c,a)},$

where $A(b,c,a) = (b \cdot c) \cdot a - b \cdot (c \cdot a)$. This implies that the set $\mathcal{R}(\mathcal{J})$ of multiplications R_a in a Jordan algebra $\mathcal{J}$ is a Lie triple system. The Lie multiplication algebra is the Lie algebra generated by $\mathcal{R}(\mathcal{J})$. This is $\mathcal{R}(\mathcal{J}) + [\mathcal{R}(\mathcal{J}), \mathcal{R}(\mathcal{J})]$. (2) implies also that every mapping $[R_a R_b]$ is a derivation in $\mathcal{J}$. Consequently every mapping $\Sigma [R_{a_i} R_{b_i}]$ is a derivation. Such derivations are called inner. Their totality $[\mathcal{R}(\mathcal{J}), \mathcal{R}(\mathcal{J})]$ is an ideal in the derivation algebra of $\mathcal{J}$.

An algebra $\mathcal{A}$ is called alternative if

(3) $\qquad a^2 b = a(ab) \ , \qquad b\, a^2 = (ba)a .$

Here we write a_R for $x \longrightarrow xa$ and a_L for $x \longrightarrow ax$. Then the defining conditions are $a_R^2 = (a^2)_R$, $a_L^2 = (a^2)_L$. Consequences of (3) are that $(ab)a = a(ba)$ and in general it can be shown

that the subalgebras generated by two elements of $\mathcal{U}$ are associative. It is immediate that if $(a,b,c) = (ab)c - a(bc)$ then this function is alternating, that is, is unchanged under even permutation and changes sign under odd permutation of the arguments. If $\mathcal{U}$ is alternative of characteristic $\neq 2$ then $\mathcal{U}$ defines a Jordan algebra $\mathcal{U}^+$ obtained by replacing the given multiplication by $a \cdot b \equiv \frac{1}{2}(ab+ba)$. In $\mathcal{U}^+$ we have $R_a = \frac{1}{2}(a_R + a_L)$. If $\mathcal{U}$ is associative and $\mathcal{J}$ is isomorphic to a subalgebra of an $\mathcal{U}^+$ then $\mathcal{J}$ is called a special Jordan algebra. It is easy to see that if $\mathcal{U}$ is alternative then $\mathcal{U}^+$ is special. In any alternative algebra one has the identities

$$[a_L b_R] = (ab)_R - a_R b_R = (ba)_L - a_L b_L$$
$$= [a_R, b_L] = b_R a_R - (ba)_R = b_L a_L - (ab)_L . \tag{4}$$

These can be used to prove that for characteristic not two

$$D_{a,b} = [a_L b_L] + [a_L b_R] + [a_R b_R] = [ab]_R - [ab]_L - 3[a_L b_R] \tag{5}$$

is a derivation. It follows that $\Sigma D_{a_i, b_i}$ is a derivation. Such derivations are called inner. Their totality is an ideal in the derivation algebra.

From now on the characteristic will be assumed different from two and all spaces and algebras will be assumed finite dimensional.

1. Jordan algebras of symmetric bilinear forms. Let $\mathcal{M}$ be a vector space over a field Φ with a non-degenerate symmetric bilinear form (x,y). We form the vector space $\mathcal{M}^* = \Phi 1 \oplus \mathcal{M}$ and define a bilinear multiplication in $\mathcal{M}^*$ such that $1 \cdot a = a = a \cdot 1$ for every a and $x \cdot y = (x,y)1$ for x, y in $\mathcal{M}$. Thus we have for α, β in Φ and x, y in $\mathcal{M}$:

(6) $\qquad (\alpha 1 + x)(\beta 1 + y) = (\alpha\beta + (x,y))1 + \alpha y + \beta x$

Evidently $a \cdot b = b \cdot a$ and a direct verification shows that the second equation in (1) holds. Hence $\mathfrak{M}^*$ is a Jordan algebra, which we shall call the Jordan algebra of the symmetric bilinear form (x,y).

Proposition 1. If $\dim \mathfrak{M} > 1$ then $\mathfrak{M}^*$ is a central simple Jordan algebra.

Proof. Let $\mathfrak{L}$ be a non-zero ideal in $\mathfrak{M}^*$ and let $\beta 1 + x$, $x \in \mathfrak{M}$, be a non-zero element of $\mathfrak{L}$. If $\beta \neq 0$ we choose $y \neq 0$ so that $(x,y) = 0$. Then $\mathfrak{L}$ contains $\beta y = (\beta 1 + x) \cdot y$. There is a z such that $(\beta y, z) = 1$. Then $\mathfrak{L}$ contains $\beta y \cdot z = 1$ and $\mathfrak{L} = \mathfrak{M}^*$. If $\beta = 0$, $x \neq 0$ and the argument just used shows that $\mathfrak{L} = \mathfrak{M}^*$. Hence $\mathfrak{M}^*$ is simple. The center of a Jordan algebra is the set of elements c such that $[R_c R_a] = 0$ for all a. We have to show that this implies that $c = \gamma 1$. It suffices to show that if $z \in \mathfrak{M}$ and $[R_z R_y] = 0$ for all y in $\mathfrak{M}$ then $z = 0$. Now if $x \in \mathfrak{M}$, $(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x,y)z - (y,z)x$. If $z \neq 0$ choose $y \neq 0$ so that $(y,z) = 0$ and x so that $(x,y) = 1$. Then $A(x,y,z) \neq 0$. This completes the proof.

Our proof shows also that every $z \in \mathfrak{M}$ is an associator. Conversely, if $a = \alpha 1 + x$, $b = \beta 1 + y$, $c = \gamma 1 + z$, $\alpha, \beta, \gamma \in \Phi$, $x, y, z \in \mathfrak{M}$ then $A(a,b,c) = A(x,y,z) = (x,y)z - (y,z)x \in \mathfrak{M}$. Hence $\mathfrak{M}$ can be characterized as the set (or space) of all the associators of elements of $\mathfrak{M}^*$.

If $a = \alpha 1 + x$, $\alpha \in \Phi$, $x \in \mathfrak{M}$ we define the (generic) trace and norm by

(7) $\qquad t(a) = 2\alpha, \qquad n(a) = \alpha^2 - (x,x)$

Then we have

(8) $\qquad a^2 - t(a)a + n(a)1 = 0.$

Write $t(a,b) = t(a\cdot b)$, $n(a,b) = \frac{1}{2}[n(a+b) - n(a) - n(b)]$.

Then

(9) $\qquad t(a,b) = 2\alpha\beta + 2(x,y),\ n(a,b) = \alpha\beta - (x,y)$

if $b = \beta 1 + y$. Both of these bilinear forms are symmetric and non-degenerate and $\mathfrak{M}$ and $\Phi 1$ are orthogonal relative to both forms. The forms reduce respectively to $2(x,y)$ and to $-(x,y)$ in the subspace $\mathfrak{M}$. We introduce the linear mapping π in $\mathfrak{M}^*$ such that

(10) $\qquad 1\pi = 1, \qquad x\pi = -x$ for $x \in \mathfrak{M}$.

This is an automorphism of period two in $\mathfrak{M}^*$ and if we write $\bar{a} = a\pi$ then

(11) $\qquad a\cdot\bar{a} = n(a)1$.

The bilinear form (x,y) leads via the Jordan algebra $\mathfrak{M}^*$ to a bilinear form $n(a,b)$ in $\mathfrak{M}^*$. Conversely, if $n(a,b)$ is any non-degenerate bilinear form in a space $\mathfrak{M}^*$ then replacing n by a suitable non-zero multiple permits us to suppose that there is an element 1 in $\mathfrak{M}^*$ such that $n(1) = 1$. Then we can write $\mathfrak{M}^* = \Phi 1 \oplus \mathfrak{M}$ where $\mathfrak{M} = (\Phi 1)^{\perp}$. If we define $(x,y) = -n(x,y)$ for x,y in $\mathfrak{M}$ then $\mathfrak{M}^*$ can be regarded as the Jordan algebra of $\mathfrak{M}$ and $n(a,b)$ becomes the form defined above by the Jordan algebra. This shows that there is nothing special about $n(a,b)$ (other than the existence of a 1 with $n(1) = 1$).

It is easy to check that

(12) $\qquad t(a\cdot b,c) = t(a,b\cdot c)$.

This shows that $R_b : x \longrightarrow x\cdot b$ is symmetric (self-adjoint) relative to the trace form $t(a,b)$. On the other hand, if $y \in \mathfrak{M}$, then

(13) $\qquad a\cdot y = \alpha y + (x,y)1 = -n(a,y)1 + n(a,1)y,$

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