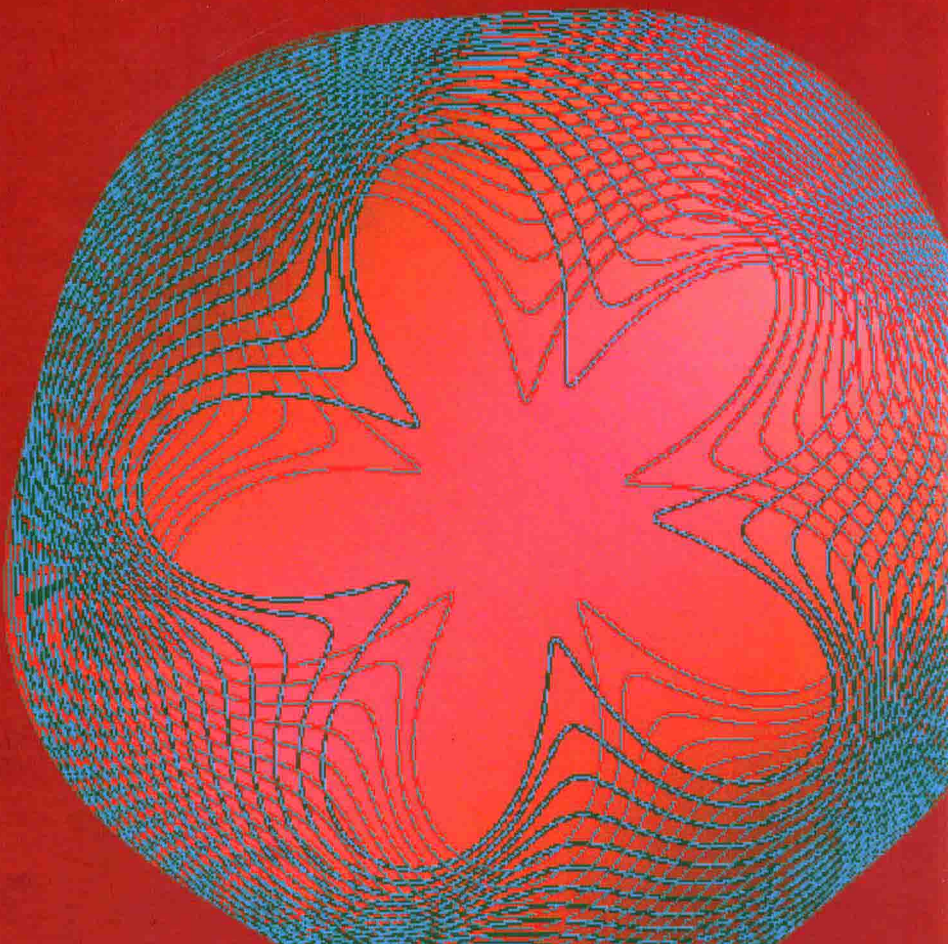
From Spinors to Quantum Mechanics

Gerrit Coddens



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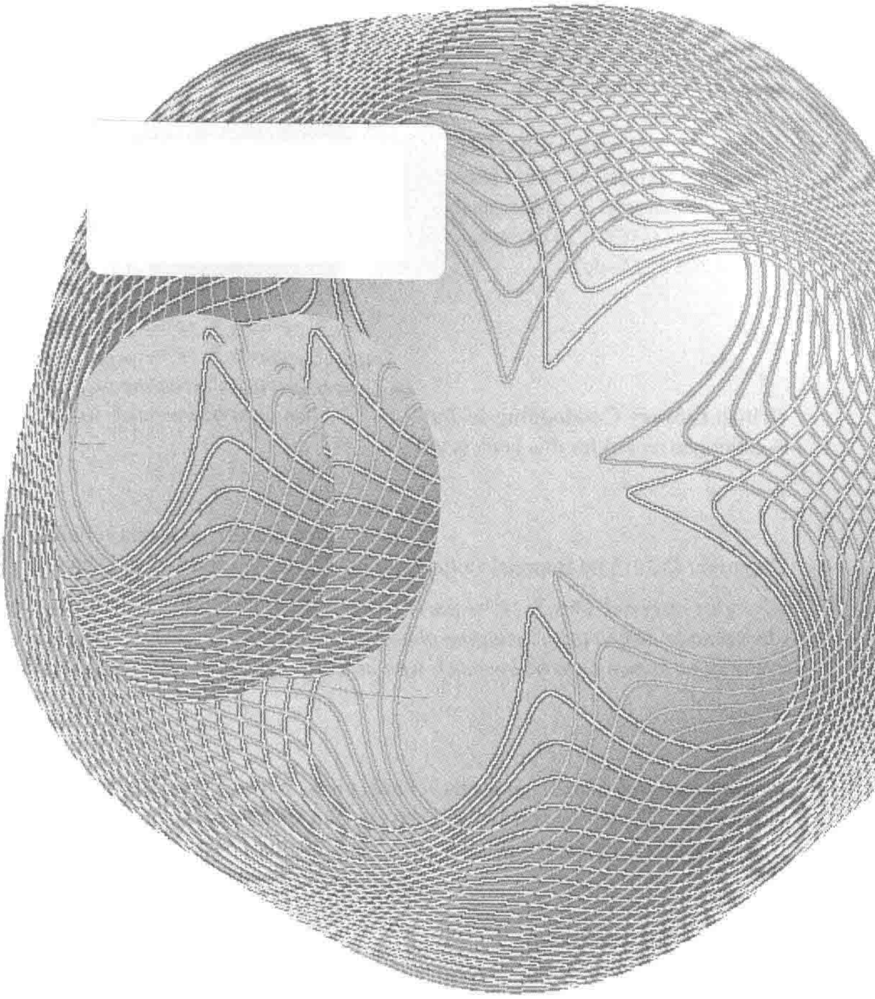

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From Spinors to Quantum Mechanics

"I came to offer thee a flower,
But thou must have all my garden,-
It is thine."
(*Rabindranath Tagore*)

To the memory of my parents
To Claude, Isabelle, Felice and Chantal
To José, Poesje, Guy and Kenzie
To Théo and Sarah

Preface

This book has a two-fold objective: giving the reader a non-conventional introduction to the representation theory for the rotation and the homogeneous Lorentz groups, that will allow them to understand these topics better, and to show how the insights gained this way can lead to a better understanding of quantum mechanics. Contrary to what a knowledgeable reader may expect on the basis of this statement, it will not be something he has seen elsewhere, neither for group theory nor for quantum mechanics.

This alternative approach to the representation theory is geometrical. Rotations are just Euclidean geometry; it should be a piece of cake. But textbooks keep it abstract, only covering the algebraic aspects of the representation theory while not explaining what the geometric counterpart of that algebra is. One can readily check that the SU(2) matrices behave like they should do. But somebody must have discovered this, based on insight, and that geometrical insight is *not* explained in textbooks. When one tries to figure it out oneself, one will have to pay an unreasonable price in amount of time spent and frustration endured. The reason for this is that a radical change in approach is required. The underlying idea is not difficult but it can take the unwary completely off guard: one has to modify the definition of a rotation as a function, by changing both its domain and range.

We are used to seeing the rotations as functions g from $\mathbb{R}^3$ to $\mathbb{R}^3$, that rotate vectors $\mathbf{r} \in \mathbb{R}^3$ to other vectors $g(\mathbf{r}) = \mathbf{r}' \in \mathbb{R}^3$. But one can see a rotation g also as a function on the group of rotations G, that transforms group elements $g_j \in G$ into other group elements $g^\circ g_j = g'_j \in G$. Instead of operating with rotations on vectors coded in the form of 3×1 column matrices (rotating vectors), one operates then with rotations on (other) rotations coded in the form of 2×1 column matrices (rotating rotations). The explicit mention of this prerequisite change in imagery is the only link that is missing; the rest is straightforward.

One can derive the representation theory for the Lorentz group following the same principles. This is more difficult because there a few more quirky mathematical twists to it. But once these have been ironed out, group theory will no longer appear as a concatenation of tedious and mysterious algebraic calculations. The algebraic quantities will have acquired a recognizable geometrical meaning, just as one recognizes a circle in the equation of a circle.

In learning quantum mechanics, one goes through the same feelings of alienation and dismay as with learning group theory. Here is this intricate set of rules with this very disorienting explanation for it. It just comes out of the blue, and it looks ever so hard to make sense of it. Much as with group theory, the reader is invited to just stick to the algebra without asking any further questions as to what it means. Certain claims, for instance that a particle cannot have a well-defined position and a well-defined momentum at the same time, render the subject even more impervious. The Hungarian philosopher Imre Lakatos summarized it wittily as follows: "*When a particle is accelerated in Brookhaven, it is not in Brookhaven*". Even more puzzling is that this is being derived from a mathematical formalism wherein the momentum and position vectors $\mathbf{p}$ and $\mathbf{r}$ appear as very well-defined quantities in the equations.

The narrative appears thus to run a bit as follows: these quantities do not exist simultaneously, but by starting from an incorrect theory based on the assumption that they *do* exist simultaneously, one can derive mathematically another, correct theory wherein they *do not* exist simultaneously. It just happens that there exists some magic that can be used to find the right starting from the wrong. It is hard to understand how this could be. Quantum mechanics is full of such mysteries.

What is proposed in this book is that the geometrical insights from group representation theory can be very helpful in making sense of quantum mechanics. It will take even some more surprising mathematical leaps, but ultimately many mysterious aspects of quantum mechanics become clear when one bases the reasoning on the true geometrical meaning of a spinor. From the results I have obtained up to now, I am convinced that this method is the only one that might permit us to eventually understand the whole of quantum mechanics. I think that this book could function as a welcome complement to anyone who wishes to obtain a better understanding of the group representation theory and of quantum mechanics.

The contents of this book have all been rethought from scratch; it has been a long and winding road. None of the results derived are novel; they have all been well known within the traditional approach for a long time. But the work cannot be assessed using a criterion of novelty of results. What counts in this book is the additional insight that can be gained from an alternative approach.

I would like to thank my employer, the Commissariat à l'Energie Atomique et aux Energies Alternatives, and my directors Guillaume Petite, Martine Soyer and Kees van der Beek for having offered me the opportunity to carry out this work, and my colleagues for their moral support. Finally I would like to thank Sébastien Ceste for his help with the figures.

Palaiseau,
Gerrit Coddens August 2011

List of Symbols

$\mathbb{1}$	unit matrix
$\otimes$	tensor product
$\boldsymbol{\nabla}$	gradient
$\Box$	d'Alembertian operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$
$\forall$	for all
$\exists$	there exists
$\exists!$	there exists a unique
$\parallel$	parallel
$\perp$	perpendicular
$*$	scalar product in space-time: $(a_{ct}, \mathbf{a}) * (b_{ct}, \mathbf{b}) = a_{ct}b_{ct} - \mathbf{a} \cdot \mathbf{b}$
$\subset$	subset
$\Rightarrow$	logical implication
$\Leftrightarrow$	logical equivalence
&	logical "and" operator
$\vee$	logical "or" operator
$\neg$	logical negation
/	not
$\mathbf{A}^\dagger$	hermitian conjugated matrix of matrix $\mathbf{A}$
$\mathbf{A}^\top$	transposed of matrix $\mathbf{A}$
A_n	alternating group of n elements (even permutations)
$a\|b$	a is replaced by b
$\{a, b, \cdots\}$	set containing $a, b, \cdots$
$\mathbf{a} \wedge \mathbf{b}$	vector product of vectors $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$
$< \mathbf{a}, \mathbf{b} >$	Hermitian in-product $\sum_{j=1}^{n} a_j^* b_j$ of $\mathbf{a} \in \mathbb{C}^n$ and $\mathbf{b} \in \mathbb{C}^n$
$A \cap B$	intersection of sets A and B
$A \cup B$	union of sets A and B
$A \backslash B$	difference of sets A and B

$\mathbf{a}\cdot\hat{\mathscr{L}}$	angular-momentum operator $a_x\hat{\mathrm{L}}_x\sigma_x + a_y\hat{\mathrm{L}}_y\sigma_y + a_z\hat{\mathrm{L}}_z\sigma_z$
$\mathbf{a}\cdot\boldsymbol{\sigma}$	notation for the 2×2 matrix $a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$
α	fine-structure constant $\alpha = \frac{q^2}{c\hbar}$
c	speed of light in vacuum
$\mathscr{C}$	light cone in Minkowski space-time $\mathbb{R}^4$
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^n$	set of n-dimensional complex vectors
$\hat{\mathrm{D}}$	$\mathbf{r}\cdot\boldsymbol{\nabla} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$
$\mathbf{D}(g)$	representation matrix of group element g
$\det(\mathbf{A})$	determinant of the matrix $\mathbf{A}$
Δ	Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
δ_{jk}	Kronecker symbol
$\hat{\mathrm{E}}$	energy operator $-\frac{\hbar}{\imath}\frac{\partial}{\partial t}$
$(\mathbf{E},\mathbf{B})$	electromagnetic field
$\mathbf{e}_j$	unit vector
ε	eccentricity of an orbit
$F(A,B)$	set of functions with domain A and values in B
φ	rotation angle
$g \in G$	group element g belongs to the group G
$g_2 \circ g_1$	abstract group operation in group $(G, \circ)$, g_2 "after" g_1
$g_{\mu\nu}$	metric tensor
γ, β	Lorentz factor and velocity parameter
γ_μ	the four Dirac matrices $\gamma_{ct}, \gamma_x, \gamma_y, \gamma_z$
$\boldsymbol{\gamma}$	the four Dirac matrices $(\gamma_{ct}, \gamma_x, \gamma_y, \gamma_z)$ in "vector" notation
h	Planck's constant
$\hbar$	$h/2\pi$
$\mathscr{I}$	isotropic cone in $\mathbb{C}^3$
J	total angular momentum, from coupling ℓ and S, $J = \ell + S$
J_z	spin quantum number for z-component of $\hat{\mathbf{J}}$
$\mathbf{L}$	Lorentz transformation matrix
ℓ	quantum number of $\hat{\mathrm{L}}$
$\hat{\mathscr{L}}$	the three operators $(\hat{\mathrm{L}}_x\sigma_x,\ \hat{\mathrm{L}}_y\sigma_y,\ \hat{\mathrm{L}}_z\sigma_z)$ in "vector notation"
L(n,$\mathbb{C}$)	linear group of $n \times n$ complex matrices
L(n,$\mathbb{R}$)	linear group of $n \times n$ real matrices
$\hat{\mathrm{L}}_z$	$-\imath\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$, sometimes just $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$
$\hat{\mathrm{L}}_z$	$-\imath\frac{\partial}{\partial\phi}$ in spherical coordinates
λ	eigenvalue or wavelength (depending on the context)

m	quantum number of $\hat{\mathrm{L}}_z$
m	relativistic mass of a moving electron
m_0	electron rest mas
m_*	modified electron rest mass (within a potential)
$\mathbb{M}_n$	Riemann manifold constructed from n copies of $\mathbb{R}^3$
μ	gyromagnetic ratio $\frac{q}{2m_0c}$
$\boldsymbol{\mu}$	magnetic dipole
$\mu_B = \hbar\mu$	Bohr magneton
$\mathbf{n}$	mathematical rotation axis specified by a unit vector $\mathbf{n}$
$\mathbb{N}$	set of positive integer numbers
ω, Ω	angular frequencies
ω_0	angular frequency in rest frame
$\hat{\mathrm{p}}_j$	momentum operator $\frac{\hbar}{\imath}\frac{\partial}{\partial x_j}$
$P_{\ell,m}$	spherical harmonic, belonging to $F(\mathbb{C}^3,\mathbb{C})$ or $F(\mathbb{R}^3,\mathbb{R})$
ψ	wave function (Schrödinger equation), spinor in SU(2) or SL(2,$\mathbb{C}$)
Ψ	2×2 (SL(2,$\mathbb{C}$)) or 4×1 (Dirac representation) spinor
ψ^c	conjugated spinor of spinor ψ in SU(2)
q	electron charge
$\mathbb{Q}$	set of rational numbers
r	radius, length of position vector $\mathbf{r}$
$\mathbf{R}$	rotation matrix
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	set of n-dimensional real vectors
$\mathbb{R}P^2$	projective space, set of directions of $\mathbb{R}^3$
$\mathbf{s}$	physical rotatation axis as specified by a unit vector $\mathbf{s} = \mathbf{e}'_z$
$\mathbf{S}$	spin vector $\frac{\hbar}{2}\mathbf{s}$
S	spin quantum number $\frac{1}{2}$, used in coupling schemes $J = \ell + S$
$S_1 \rightsquigarrow S_2$	the calculations on set S_1 lead to the same result as on set S_2
SL(2,$\mathbb{C}$)	two-dimensional complex special linear group
S_n	symmetric group of n elements (permutations)
SO(3)	three-dimensional special orthogonal group (rotations in $\mathbb{R}^3$)
SU(2)	two-dimensional special unitary group (rotations in $\mathbb{R}^3$)
S_z	spin quantum number for z-component $\pm\frac{1}{2}$ of $\hat{\mathbf{S}}$
$\boldsymbol{\sigma}$	the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ in "vector" notation
σ_j	the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$
ς	curvilinear path length

(θ, ϕ)	spherical coordinates
τ	proper time
$\mathbf{V}^\star$	$ct\mathbb{1} - \mathbf{r}\cdot\boldsymbol{\sigma}$ for $\mathbf{V} = ct\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}$
$(V, \mathbf{A})$	electromagnetic four-potential
(ξ_0, ξ_1)	components of a spinor of SU(2)
$(\dot{\xi}_0, \dot{\xi}_1)$	dotted spinor
Y	icosahedral group
$Y_{\ell,m}$	complex spherical harmonic as a function belonging to $F(\mathbb{R}^2, \mathbb{C})$
$\mathbb{Z}$	set of integer numbers
ζ	$\zeta = \xi_1/\xi_0$ (only used in the stereographic projection)

Contents