IRMA Lectures in Mathematics
and Theoretical Physics 26

Handbook of Teichmüller Theory
Volume V

Athanase Papadopoulos
Editor

European Mathematical Society

Handbook of Teichmüller Theory

Volume V

Athanase Papadopoulos
Editor

European Mathematical Society

Editor:

Athanase Papadopoulos
Institut de Recherche Mathématique Avancée
CNRS et Université de Strasbourg
7 Rue René Descartes
67084 Strasbourg Cedex
France

2010 Mathematics Subject Classification: Primary 30-00, 32-00, 57-00, 32G13, 32G15, 30F60;
Secondary 11F06, 11F75, 14D20, 14H15, 14H60, 14H55, 14J60, 20F14, 20F28, 20F38, 20F65, 20F67, 20H10, 22E46, 30-03, 30C62, 30F20, 30F25, 30F10, 30F15, 30F30, 30F35, 30F40, 30F45, 32-03, 32S30, 37-99, 53A35, 53B35, 53C35, 53C50, 53C80, 53D55, 53Z05, 57M07, 57M20, 57M27, 57M50, 57M60, 57N16.

ISBN 978-3-03719-160-6

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum SEW A27
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the authors' TEX files: le-tex publishing services GmbH, Leipzig, Germany
Printing and binding: Beltz Bad Langensalza GmbH, Bad Langensalza, Germany
∞ Printed on acid free paper
9 8 7 6 5 4 3 2 1

IRMA Lectures in Mathematics and Theoretical Physics 26

Edited by Christian Kassel and Vladimir G. Turaev

Institut de Recherche Mathématique Avancée
CNRS et Université de Strasbourg
7 rue René Descartes
67084 Strasbourg Cedex
France

IRMA Lectures in Mathematics and Theoretical Physics
Edited by Christian Kassel and Vladimir G. Turaev

This series is devoted to the publication of research monographs, lecture notes, and other material arising from programs of the Institut de Recherche Mathématique Avancée (Strasbourg, France). The goal is to promote recent advances in mathematics and theoretical physics and to make them accessible to wide circles of mathematicians, physicists, and students of these disciplines.

Previously published in this series:

6 *Athanase Papadopoulos*, Metric Spaces, Convexity and Nonpositive Curvature
7 Numerical Methods for Hyperbolic and Kinetic Problems, *Stéphane Cordier, Thierry Goudon, Michaël Gutnic and Eric Sonnendrücker* (Eds.)
8 AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries, *Oliver Biquard* (Ed.)
9 Differential Equations and Quantum Groups, *D. Bertrand, B. Enriquez, C. Mitschi, C. Sabbah and R. Schäfke* (Eds.)
10 Physics and Number Theory, *Louise Nyssen* (Ed.)
11 Handbook of Teichmüller Theory, Volume I, *Athanase Papadopoulos* (Ed.)
12 Quantum Groups, *Benjamin Enriquez* (Ed.)
13 Handbook of Teichmüller Theory, Volume II, *Athanase Papadopoulos* (Ed.)
14 *Michel Weber*, Dynamical Systems and Processes
15 Renormalization and Galois Theory, *Alain Connes, Frédéric Fauvet and Jean-Pierre Ramis* (Eds.)
16 Handbook of Pseudo-Riemannian Geometry and Supersymmetry, *Vicente Cortés* (Ed.)
17 Handbook of Teichmüller Theory, Volume III, *Athanase Papadopoulos* (Ed.)
18 Strasbourg Master Class on Geometry, *Athanase Papadopoulos* (Ed.)
19 Handbook of Teichmüller Theory, Volume IV, *Athanase Papadopoulos* (Ed.)
20 Singularities in Geometry and Topology. Strasbourg 2009, *Vincent Blanlœil and Toru Ohmoto* (Eds.)
21 Faà di Bruno Hopf Algebras, Dyson–Schwinger Equations, and Lie–Butcher Series, *Kurusch Ebrahimi-Fard and Frédéric Fauvet* (Eds.)
22 Handbook of Hilbert Geometry, *Athanase Papadopoulos and Marc Troyanov* (Eds.)
23 Sophus Lie and Felix Klein: The Erlangen Program and Its Impact in Mathematics and Physics, *Lizhen Ji and Athanase Papadopoulos* (Eds.)
24 Free Loop Spaces in Geometry and Topology, *Janko Latschev and Alexandru Oancea* (Eds.)
25 *Takashi Shioya*, Metric Measure Geometry. Gromov's Theory of Convergence and Concentration of Metrics and Measures

Volumes 1–5 are available from De Gruyter (www.degruyter.de)

Foreword

Teichmüller theory is a vast subject which encompasses ideas and techniques from complex analysis, hyperbolic geometry, topology, partial differential equations, algebraic geometry, Kähler geometry, geometric group theory, representation theory, dynamical systems, number theory and from other fields, with applications in mathematical physics (string theory, conformal and topological field theories, two-dimensional gravity, etc.) and, more recently, in biology. Besides the "classical" Teichmüller theory, there is a "quantum Teichmüller theory", a "discrete Teichmüller theory", and a "higher Teichmüller theory."

This Handbook project arouse from the desire of collecting in a systematic way a set of surveys covering all these theories and making them easily accessible to researchers and students. The result is due to the effort of many people, and above all the hundred authors who contributed to the various volumes which already exist in print. Let them all be thanked here.

This collective work reflects the fact that Teichmüller theorists form a community. At the same time, skimming through all these pages gives a profound feeling of unity in mathematics.

The present volume is divided into the following four parts:

- Part A. The metric and the analytic theory, 5
- Part B. The group theory, 4
- Part C. Representation theory and generalized structures, 3
- Part D. Sources, 2

The number after the name of each part indicates that it is a sequel to a part carrying the same name (with a different numbering) in a previous volume of the Handbook.

Athanase Papadopoulos
Strasbourg and Tokyo, November 2015

Contents

Part D. Sources

Introduction to Teichmüller theory, old and new, V

Athanase Papadopoulos

Contents

1 Introduction

Teichmüller theory is a broad subject whose roots lie in the work of Bernhard Riemann who introduced in his doctoral dissertation (1851) an equivalence relation on the set of Riemann surfaces and stated that the number of (complex) "moduli" for the set of equivalence classes, in the case of a Riemann surface of genus $g \geq 2$, is $3g - 3$. Several theories developed by major mathematicians during the two hundred years that followed this assertion were motivated to a large extent by the effort to give a precise meaning to this moduli count. We call this problem the "Riemann moduli problem." It was given a huge impetus by Oswald Teichmüller who wrote several papers between 1937 and 1943, constructing a space (that was called later on "Teichmüller space") equipped with various structures, including a Finsler metric and a complex-analytic structure of the right dimension, thus giving to Riemann's count the status of a complex dimension and in this sense solving Riemann's problem. At the same time, Teichmüller's papers paved the way to several geometrical and analytical results on spaces of equivalence classes of Riemann surfaces and to the develoment of several theories by major mathematicians of the twentieth century, including Ahlfors, Bers, Weil, Grothendieck, Thurston, and there are many others.

The present volume, like the preceding ones in this series, contains a collection of surveys on classical Teichmüller theory and on some of its modern developments. Certain topics that are considered were already treated from a different point of view in previous volumes. As a matter of fact, some of the chapters may be considered as sequels to chapters in the preceding volumes, surveying some particular subject in more detail or from a new perspective. For instance, Chapter 2, which consists of a collection of commented open problems on Thurston's metric on Teichmüller space, is a natural sequel to Chapter 2 of Volume I which is a survey of that metric, written eight years ago, and which already contains some open problems. Some progress on that theory has been made during these years, and one of the main advances is the characterization of the horofunction boundary and the isometry group of this metric. These results are contained in Chapter 7 of Volume IV of this Handbook. Some of the material in Chapter 3 on the Meyer cocycle and Meyer functions was already mentioned in Chapter 6 of Volume II, whose subject is the cohomology of the mapping class group. Likewise, Chapter 5 which is a survey of the Johnson homomorphisms for automorphisms of free groups and more general groups, is a sequel to a section in Chapter 7 of Volume I where the Johnson homomorphisms for mapping class groups are studied. Chapter 6 on the geometry and dynamics on character varieties is related to Chapter 13 of Volume IV on higher Teichmüller theory. There are several other connections between chapters in this volume and other ones in preceding volumes.

We present in some detail the topics treated this volume. They are grouped in four parts.

2 Part A, The metric and the analytic theory, 5

Part A of this volume includes two chapters. They concern identities on hyperbolic surfaces and the Thurston metric.

2.1 Identities on hyperbolic surfaces

Chapter 1, written by Martin Bridgeman and Ser Peow Tan, concerns some remarkable identities on the length spectrum of hyperbolic surfaces (in particular lengths of simple closed curves and of properly embedded arcs) and generalizations to higher-dimensional manifolds. The first such identities were discovered at the beginning of the 1990s by Greg McShane and Ara Basmajian. The theory was further developed by several authors who transformed, extended, generalized these identities and gave them new proofs.

McShane's identity, in its original form (1991), says that for any complete finite-volume hyperbolic structure on a once-punctured torus, we have

$$\sum_{\gamma} \frac{1}{1+e^{l(\gamma)}} = \frac{1}{2}$$

where the sum is over all simple closed geodesics γ, $l(\gamma)$ being its length.

Basmajian's identities (1993) hold more generally for hyperbolic manifolds of finite volume with non-empty geodesic boundary, in any dimension ≥ 2. They concern the *ortholength spectrum*, that is, the set of lengths of properly immersed geodesic arcs which make right angles with the boundary. Such arcs are called *orthogeodesics*. Basmajian proved that a totally geodesic surface in a hyperbolic manifold M can be decomposed into embedded discs which are in one-to-one correspondence with the orthogeodesics of the manifold with boundary obtained by cutting M along that surface. He deduced an identity relating the volume of that surface and the ortholength spectrum of M. In the case where the manifold is a surface S with non-empty boundary ∂S, Basmajian's identity becomes:

$$\text{Length}(\partial S) = \sum_{l \in L_S} 2 \log \left(\coth \frac{l}{2} \right)$$

where the sum is over the ortholength spectrum L_S of S.

More recently, in 2009, Bridgeman showed that the unit tangent bundle $T_1(S)$ of a hyperbolic surface S with nonempty boundary ∂S can be decomposed into certain "drum-like" pieces which are in correspondence with the orthogeodesics of S. He obtained the following identity for the ortholength spectrum:

$$\text{Vol}(T_1(S)) = 2\pi \text{Area}(S) = \sum_{l \in L_S} 4\mathcal{R} \left(\text{sech}^2 \frac{l}{2} \right),$$

where $T_1(S)$ is equipped with its canoncial volume form, where the sum is again over the ortholength spectrum L_S, and where $\mathcal{R}$ is Rogers' dilogarithm function.

Soon after Bridgeman's result, Bridgeman–Kahn and Calegari gave identities that are valid in any dimension and generalize Bridgeman's identity.

These identities have several applications. The most spectacular application of McShane's identity is probably the one discovered by Mirzakhani who extended this identity to bordered surfaces and used it to show that the Weil–Petersson volumes of moduli spaces of such surfaces are polynomial functions of the lengths of the boundary components. The constant terms that appear in the polynomials she discovered are the Weil–Petersson volumes of the moduli spaces of complete hyperbolic surfaces of genus g with n punctures and with no boundary components. Furthermore, Mirzakhani gave recursive formulae for the other coefficients of the polynomials, which turned out to be the intersection numbers of the so-called tautological classes of the Deligne–Mumford compactification of moduli space. As a consequence, she obtained a completely new proof of the Witten conjecture (1991, proven by Kontsevich in 1992). We recall that this conjecture proposes a recursive formula for the tautological classes, stating that a certain generating function for their intersection numbers satisfies a series of KdV differential equations.

In another direction, Luo and Tan, motivated by the works of McShane, Mirzakhani and Bridgeman, obtained an identity that generalizes the McShane identity, which is valid for surfaces with or without boundary and which involves the dilogarithms of the lengths of the simple closed geodesics in all 3-holed spheres and 1-holed tori embedded in the surface.

In Chapter 1 of this volume, the authors survey the above identities, making connections between them and proposing a unified approach for the proofs. This involves the definition, for the hyperbolic manifold M whose length (respectively ortholength, etc.) spectrum is investigated, of a certain set X associated to M, equipped with a finite measure μ, and the study of a measure-theoretic decomposition of X into countably many disjoint subspaces X_i of finite non-zero measure and a subspace Z which has zero measure. Typically, X may be the boundary of M, or the set of geodesics in M embedded in the unit tangent bundle and equipped with the associated geodesic flow, or a set of random geodesics, and there are other possibilities. The general identity that the authors obtain has the form of a summation formula

$$\mu(X) = \sum_i \mu(X_i)$$

which the authors call a *tautological identity*.

The subject of identities involving lengths of geodesics is very active now, and the authors in Chapter 1 mention works of Bowditch, Akiyoshi–Sakuma–Miyachi, Labourie–McShane, Luo–Tan, Hu, Hu–Tan–Zhang, Kim–Tan, Tan–Wong–Zhang and others, indicating at some places connections between the various results obtained.

2.2 Problems on Thurston's metric

Chapter 2, by Weixu Su, consists of a set of commented problems on Thurston's asymmetric metric on Teichmüller space. This metric was already surveyed in Chapter 2 of Volume I of this Handbook, written by Théret and the author of this introduction. That chapter contained a commented list of 13 problems. Since then, some of these problems were solved, new results were obtained, and it seemed natural to us to include a new list of problems, especially since this metric has been recently the subject of intense investigation. The chapter contains 46 problems, divided into 5 sections:

(1) Infinitesimal properties;

(2) Geodesics;

(3) Generalizations;

(4) Infinitely-generated Fuchsian groups of the first kind;

(5) Other general questions.

The first section concerns the Finsler infinitesimal norm and its influence on the large-scale properties, like quasi-isometries and the behavior of stretch lines and more general geodesics. The question of the relation of this infinitesimal norm with the complex structure is also addressed. Section (2) contains questions concerning the totality of geodesics joining two points, geodesic flows, the number of closed geodesics in moduli space of length $\leq R$ and the (equi)-distribution of these geodesics, the existence of dense geodesics in moduli space, and the asymptoticity question (when do geodesic rays stay at a bounded distance from each other?). Questions on the group of quasi-isometries are again addressed. Section (3) contains questions on symmetrizations of Thurston's metric (the so-called length-spectrum metric and others) and other metric generalizing this metric, like the arc metric for surfaces with boundary. Questions concerning the global and local properties, the boundaries and the isometries of these metrics are addressed. The questions in Section (4) concern generalizations to length spectra of surfaces of infinite type, namely, whether there exist quasiconformal deformations for which the lengths of all simple closed curves do not increase. In the case of a positive response, this would mean that the same formula for the Thurston metric, if it is used in the case of a surface of infinite type, does not define a metric. The same question is addressed for extremal length instead of hyperbolic length. Related questions concern the critical exponents of infinitely-generated Fuchsian groups of the first kind. The last section contains some questions related to generalizations of the Thurston metric, including convex combinations of Thurston's metric and it reverse, and metrics on spaces of n-tori.

This problem list contained in Chapter 2 is an updated version of a list that was compiled after a workshop at the American Institute for Mathematics (Palo Alto) whose title was "Lipschitz metric on Teichmüller space." The problems were contributed by several people.

The study of Thurston's metric became recently an active field of research, and several individuals and groups of researchers are presently working on problems related to that metric. However, it is fair to say that the most interesting results concerning this metric are still, by far, those contained in the first draft of Thurston, written in 1985.

3 Part B. The group theory, 4

Part B of this volume contains three chapters. They concern mapping class groups, and in particular their cohomology.

3.1 The Meyer cocycle and Meyer functions

The cohomology of the mapping class group Γ_g of an orientable closed surfaces of genus g with rational coefficients is isomorphic to the cohomology of the Riemann moduli space of that surface. Thus, studying properties of the cohomology of the mapping class group is also a way of studying properties of Riemann's moduli space.

In 1982, Harer showed that the second cohomology group $H^2(\Gamma_g, \mathbb{Z})$ of Γ_g is isomorphic to $\mathbb{Z}$, for all $g \geq 3$. A nonzero element τ_g in that group was already highlighted by Werner Meyer in 1973 and is now called the *Meyer signature cocycle*, or more simply the *Meyer cocycle*. We start by recalling a few facts about it.

The Meyer cocycle is an integer-valued 2-cocycle that appeared in Meyer's work as a characteristic class of surface bundles over surfaces. The word "signature" refers here to the signature of surface bundles as 4-manifolds. We recall that the signature of a compact oriented 4-manifold is defined as the signature of the intersection form on the second cohomology group. It is interesting for Teichmüller theorists to know that Meyer's construction of the signature cocycle in the setting of fibered 4-manifolds uses the decomposition of the base surface into pairs of pants. The Meyer cocycle can also be obtained by pulling back by the homology representation $\Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z})$ a group 2-cocycle on the Siegel modular group $\mathrm{Sp}(2g, \mathbb{Z})$ which represents the signature class in $H^2(\mathrm{Sp}(2g, \mathbb{Z}), \mathbb{Z})$.

The intersection form of a 4-manifold is a powerful invariant. By a theorem attributed to Whitehead and Milnor, two simply-connected 4-manifolds are homotopy equivalent if and only if they have isomorphic intersection forms. This intersection form was used by Freedman in his classification of simply-connected 4-manifolds. Finally, we recall that the Hirzebruch signature theorem reduces the computation of the signature to that of the first Pontriaguine class.

Meyer studied the signature of the total space of any S_g-bundle over an oriented closed surface of any genus $g \geq 1$. He showed that for $g \leq 2$ the signature is zero. For $g \geq 3$, he showed that this signature is a multiple of 4. He also showed that for any $g \geq 4$ and for any integer n which is a multiple of 4, there exists an S_g-bundle whose signature is n. Turaev, in 1987, introduced a cocycle which turned out to be identical to Meyer's signature cocycle, and he made relations with the Maslov index in symplectic geometry. It is also known that the Meyer signature cocycle is proportional to the first Mumford–Morita–Miller class and to the Weil–Petersson 2-form (works of Atiyah and of Wolpert).

The Meyer functions are related to the Meyer cocycle. These functions play an important role in the study of the cohomology of the mapping class group Γ_g. They are defined in each genus $g \geq 1$, as secondary invariants. Meyer showed that for $g = 1$ (where $\Gamma_1 \simeq \mathrm{SL}(2, \mathbb{R})$) and for $g = 2$, there exists a unique $\mathbb{Q}$-valued cochain $\phi_g \colon \Gamma_g \to \mathbb{Q}$ whose coboundaries are respectively the Meyer signature cocycles τ_1 and τ_2. He also gave an explicit form for ϕ_1. Atiyah gave several geometric interpretations of the Meyer functions. He established relations of the value of the function ϕ_1 at a hyperbolic element of $\mathrm{SL}(2, \mathbb{Z})$ with several invariants, including the Hirzebruch signature defect, the Shimizu L-function, and the Atiyah–Patodi–Singer η invariant of signature operators of Riemannian 3-manifolds. Morita and Morifuji established relations with the Casson invariant of homology 3-spheres.

In Chapter 3 of this volume, Yusuke Kuno describes these developments as well as analogues for higher genera and higher dimensions. He also mentions the so-called local signature theory. This is based on the fact that given a closed oriented 4-manifold M fibered over a surface S, its signature is, under some conditions, localized at finitely many singular fibers of the fibration $f \colon M \to S$. The Meyer functions turn out to be useful for approaching this problem. Kuno also mentions works of Y.

Matsumoto and E. Horikawa on this subject. Matsumoto gave the first examples of computation of local signatures of fibered 4-manifolds, in genus 1 and then in genus 2. His work uses the Meyer functions ϕ_1 and ϕ_2. Horikawa considered local signature in the setting of algebraic geometry. The higher-dimensional analogue of the Meyer function ϕ_2 was also studied by Iida, who called this function the *Meyer function for smooth theta divisors*. The author of Chapter 3 also reviews his recent extension of Matsumoto's work in the setting of projective varieties.

Let us note that the Meyer cocycle was already considered in Chapter 6 of Volume II of this Handbook, by Nariya Kawazumi, and that an important class of fibered 4-manifolds, the Lefschetz fibrations, are surveyed in Chapter 7 of Volume III, by Mustafa Korkmaz and András Stipicz.

3.2 The Torelli–Johnson–Morita theory

Chapter 4, by Nariya Kawazumi and Yusuke Kuno, is a survey on the so-called *Torelli–Johnson–Morita theory*, from a new point of view which uses heavily the Goldman Lie algebra. This theory is motivated by the study of the Torelli group, based on the Johnson homomorphism and its generalization by Morita who introduced the notion of k-th order Johnson homomorphism, the k-th component of an injective graded Lie algebra homomorphism.

We recall that the first Johnson homomorphism was defined by Dennis Johnson, in his systematic study of the Torelli group which he started in the late 1970s. He introduced a homomorphism, denoted by τ, which is useful in identifying the abelianization of that group. This homomorphism is now called the first Johnson homomorphism. Motivated by Johnson's work, Morita, in 1993, generalized the Johnson homomorphism to a sequence of homomorphisms defined on a series of subgroups of the mapping class group into certain abelian subgroups. Morita's homomorphisms are called higher degree Johnson homomorphisms. It is also natural to define Johnson homomorphisms in the setting of automorphisms of free groups. Morita revived early work on this subject done by Andreadakis, in 1965. This topic is surveyed in Chapter 5 of the present volume.

The authors in Chapter 4 of the present volume consider an oriented compact surface $S = S_{g,1}$ of genus $g \geq 0$ with one boundary component, with $\Gamma_{g,1}$ its mapping class group relative to the boundary (all homeomorphisms and isotopies fix the boundary pointwise), and $\mathcal{I}_{g,1}$ its Torelli group, i.e. the subgroup of $\Gamma_{g,1}$ of elements that act trivially on the homology.

The Johnson filtration is a central filtration of $\mathcal{I}_{g,1}$ whose definition uses the action of $\Gamma_{g,1}$ on the fundamental group of S. There is a graded Lie algebra homomorphism whose k-th component is called the k-th Johnson homomorphism whose target is the infinite-dimensional Lie algebra of symplectic derivations of Lie type in the sense of Kontsevich. In fact, this graded Lie algebra homomorphism was defined by Morita by the end of the 1990s, and independently by Kontsevich. Its image is called the Johnson image. To get a characterization of this image is an important question in the study of the Torelli group $\mathcal{I}_{g,1}$.

In 1986, Goldman introduced a Lie bracket on the free $\mathbb{Z}$-module of homotopy classes of oriented loops on an oriented surface. The resulting Lie algebra is called the Goldman Lie algebra. The main object of Chapter 4 is to show how this Lie algebra appears in the Torelli–Johnson–Morita theory. This is based on the observation that the Goldman Lie algebra acts on the fundamental group of the surface by derivations. To make the relation with the action of the mapping class group by automorphisms, the authors construct completions of the Goldman Lie algebra and of the fundamental group. They introduce a morphism they call the "geometric Johnson homomorphism,", whose graded quotients give Morita's k-th Johnson homomorphisms. The geometric Johnson homomorphism turns out to be an intrinsic and choice-independent version of a map defined by Massuyeau in 2012 and called the "total Johnson map."

Turaev, in 1991, discovered a Lie cobracket on a certain quotient of the Goldman Lie algebra and he showed that this quotient algebra has the structure of a Lie bialgebra. This Lie bialgebra is called the Goldman–Turaev bialgebra, and it is also a central object of study in this survey. The Turaev cobracket induces a map from the target of the geometric Johnson homomorphism. The authors show that this gives a constraint on the Johnson image. They also mention relations with recent works of Turaev–Massuyeau and Church on related subjects. The chapter contains an overview of several aspects of the theory including the known results about the Johnson image, and on the Goldman–Turaev Lie bialgebra. The authors also discuss the extension theory of the Johnson homomorphism to the mapping class group. They also survey in detail the theory of (generalized) Dehn twists from this point of view, introduced by Kuno and extensively studied by Kawazumi and Kuno, showing that this generalized Dehn twist action on the completion of the fundamental groupoid of the surface has a canonical logarithm and specifying it as an element of the completion of the Goldman Lie algebra.

3.3 The Johnson homomorphisms

In Chapter 5, Takao Satoh surveys the theory of the Johnson homomorphisms in the setting of automorphisms of free groups. These are in some sense generalizations of the Johnson homomorphisms for mapping class groups. It is a very well known fact now that there are close relations and analogies between mapping class groups and automorphisms of free groups. We recall in this respect that the mapping class group of a compact oriented surface $S_{g,1}$ of genus g with one boundary component is embedded in the outer automorphism group of its fundamental group, which is a finitely generated free group.

The theory developed in this chapter may be considered as an instance of the analogies between the theory of mapping class groups and that of the (outer) automorphisms of free groups.

As a matter of fact, the Johnson homomorphisms can be defined for a general class of groups that contains both groups, and they are useful in the study of the graded quotients of a certain descending filtration of the automorphism group of such