Differential Equations

Second Edition

JOHN A. TIERNEY

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Preface

This second edition employs the same approach as the original text and is based on the same underlying philosophy. It is intended primarily for a sophomore or a junior level one-semester introductory course in differential equations. Addressed to students who have completed a basic calculus sequence, it is not too difficult for a good student who has completed two semesters of calculus. The level of mathematical sophistication is appropriate for average students whose interests lie in mathematics, science, or engineering. It is also suitable for those who expect to encounter differential equations in such diverse fields as economics, biology, medicine, or demography.

Although theory is not slighted, the main emphasis is on applications. These are taken not only from the physical sciences but also from the life sciences, the social sciences, and geometry. A perusal of the Table of Contents gives a fairly specific indication of the variety of applications covered. Chapters 3 and 5 are devoted entirely to applications; other chapters include applications whenever appropriate. Few applications receive long, detailed treatment since time limitations in most courses render such an approach impractical. Attention is focused on those applications that are readily understood by students who do not have specialized knowledge of the fields from which applications are selected. More applications are included than can be covered in the usual course. This permits flexibility and allows the instructor to consider the interests, preparation, and mathematical maturity of the students.

Various other options are available. Chapter 7 on the Laplace transform and Chapter 10 on partial differential equations may be omitted, treated lightly, or covered completely. Sections 4.8, 4.9, 4.11, 6.2, 9.2, and 9.7 through 9.10 are not essential to the main development and may be omitted or given minor emphasis in a short course.

The main objective of the textbook is to present simply and clearly the important concepts of the theory of differential equations and to show how differential equations are used in mathematical models describing real-life situations. This is accomplished by using simple yet modern terminology and notation, including an unusually large number of examples, and presenting extensive, carefully graded problem lists. Numerous expositions illustrate the manner in which reasonable physical assumptions lead to differential equations governing concrete situations. The text emphasizes that the same mathematical model often governs many seemingly unrelated physical situations.

The importance of numerical methods of solving differential equations is stressed. Although the computer's importance is emphasized, the material is developed without assuming that the student has access to a high-speed computer. The basic objective of Chapter 9 is to give the student an idea of what is important in numerical solutions and to present a few simple numerical methods.

The existence and the uniqueness of solutions are considered, with illustrations. This is accomplished without including the numerous proofs that would be essential in a more theoretical development. Students are encouraged to seek properties of solutions of differential equations without obtaining explicit solutions. Geometric and physical interpretations are exploited in this direction.

Numerous references are given at the end of each chapter. These furnish sources for additional proofs, more detailed theoretical approaches, and expanded treatments of applications.

This revision differs from the first edition in several respects. In Section 2.4 an example is presented in which the function Q in the differential equation $y' + P(x)\, y = Q(x)$ is discontinuous at a point $x = a$ inside the interval under consideration. Section 2.7 includes three graphs of direction fields obtained using computer graphics. These graphs will give students an awareness of the manner in which a computer, without actually solving a differential equation, can furnish information about properties of the solutions of the equation. Section 2.7 also includes several examples illustrating the Peano existence theorem and the Picard uniqueness theorem.

Chapter 4 on Linear Differential Equations has been expanded considerably. The Wronskian in Section 4.3 is given more prominence, the topic Reduction of Order is treated separately in Section 4.5, and in Section 4.6 Homogeneous Second-Order Linear Differential Equations and the Euler Identity receive more detailed treatments. The chapter contains two new sections, Section 4.10 on Higher-Order Linear Equations and Section 4.11 on the Euler Equation.

Section 6.2 on Plane Autonomous Systems, Critical Points, and Stability is new. In courses where time permits, this section will give students a better understanding of the qualitative aspects of the theory of differential equations.

In Section 8.3 the Method of Frobenius has been expanded slightly. Chapter 9 on Numerical Methods now contains separate sections on Runge-Kutta methods and the Classical Runge-Kutta Method. Section 10.7 includes two illustrations based on computer graphic treatments of the Heat Equation.

In addition to these specific changes and additions, a number of general modifications have been incorporated. To render the overall presentation more instructive, interesting, and challenging, numerous illustrations, drawings, and problems have been added. The exposition has been modified in various places in an effort to achieve greater clarity.

It is our hope that this new edition will give students an understanding and appreciation of the important role that differential equations play in modern life.

I am indebted to Professor Mahlon F. Stilwell and Marion G. Tierney for their careful reading of the manuscript and their many highly valuable suggestions.

I also express my appreciation to Raeia Maes for coordinating the production of this text.

Finally, I am deeply grateful to Allyn and Bacon editors Gary Folven and Carol Nolan-Fish. Their constant encouragement and assistance have been invaluable.

J.A.T.

Contents

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Preliminary Concepts

1.1 Introduction

In the latter half of the seventeenth century, Newton and Leibniz systematized the calculus into a unified body of mathematical knowledge. Their exploitation of the fact that differentiation and integration are inverse processes led to the development of the theory of differential equations [abbreviated DE henceforth, for "differential equation(s)"]. Mathematicians and scientists were quick to realize that many basic physical laws were best expressed by equations involving not only the underlying variables but also their derivatives or instantaneous rates of change. Interest in the theory and application of DE has persisted to the present day. Efforts to resolve various theoretical questions concerning DE have resulted in the enrichment of mathematical analysis, the study of infinite processes. Investigators continue to discover new applications of DE, not only in the physical sciences but also in such diverse fields as biology, physiology, medicine, statistics, sociology, psychology, and economics. Both theoretical and applied DE are active fields of current research.

Definition A differential equation is an equation involving unknown functions and their derivatives.

If the functions are real functions of one real variable, the derivatives occurring are ordinary derivatives, and the equation is called an *ordinary* DE.

If the functions are real functions of more than one real variable, the derivatives occurring are partial derivatives, and the equation is called a *partial* DE. When we refer to an equation as a DE, we shall mean an ordinary DE. Until we state otherwise, we shall restrict ourselves to DE involving a single unknown function.

Definition By a solution of the DE

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \tag{1.1}$$

we mean a real function f, denoted by $y = f(x)$, defined on a set S of real numbers, where S is the union of nonoverlapping intervals such that

$$F(x, f(x), f'(x), f''(x), \ldots, f^{(n)}(x)) \equiv 0$$

for all x in S.

If x_0 is a left (right) endpoint of an interval in S, derivatives on the right (left) of f at x_0 are intended. NOTE: The variables x and y need not appear in (1.1); however, at least one derivative must appear if (1.1) is to be termed a DE.

EXAMPLE 1. The function defined by $y = f(x) = e^{2x} \equiv \exp(2x)$ is a solution of the DE $y' - 2y = 0$ on $(-\infty, +\infty)$, since $2e^{2x} - 2(e^{2x}) \equiv 0$ for all x. The function defined by $y = g(x) = e^{2x}$ having domain $[0, 1]$ is also a solution of $y' - 2y = 0$.

In many applications we seek a solution of a DE where the domain of the solution is a specified interval. For example, if t denotes time in a DE, a solution is often sought on the interval $t \geq 0$. If no domain is specified, we seek a solution (or solutions) having the largest possible domain, consisting of one or more intervals.

EXAMPLE 2. The function defined by $y = \sqrt{1 - x^2}$ is a solution of the DE $yy' + x = 0$ on $(-1, 1)$, since

$$\sqrt{1 - x^2}\left(\frac{-2x}{2\sqrt{1 - x^2}}\right) + x \equiv 0$$

for all x in $(-1, 1)$. Note that y is undefined on $(-\infty, -1) \cup (1, +\infty)$, and that y' is undefined at $x = +1$ and at $x = -1$.

EXAMPLE 3. Find solutions of the DE $y' = x^{-2}$.

Solution: Since $d/dx(-x^{-1} + C) = x^{-2}$ for all $x \neq 0$, then $y = -x^{-1} + C$ defines a family of solutions on $S = (-\infty, 0) \cup (0, +\infty)$, where C is an arbitrary constant. Thus, the DE has an infinite number of solutions, each having domain S.

EXAMPLE 4. Find solutions of $y' = \dfrac{x}{\sqrt{x^2 - 1}} - \dfrac{x}{\sqrt{9 - x^2}}$.

Solution: Since

$$\frac{d}{dx}(\sqrt{x^2-1}+\sqrt{9-x^2}+C)=\frac{x}{\sqrt{x^2-1}}-\frac{x}{\sqrt{9-x^2}}$$

then

$$y=\sqrt{x^2-1}+\sqrt{9-x^2}+C$$

defines a family of solutions, each with domain $S=(-3,-1)\cup(1,3)$.

Example 5. Find solutions of $y'=\frac{3\sqrt{x}}{2}$.

Solution: Since

$$\frac{d}{dx}(x^{3/2}+C)=\frac{3\sqrt{x}}{2}$$

then $y=f(x)=x^{3/2}+C$ defines a family of solutions, each having domain $S=[0,+\infty)$. By $f'(0)=0$ we mean the derivative on the right of f at $x=0$.

Example 6. The DE $(dy/dx)^2+x^2y^2+1=0$ has no real-valued solution, since the left member is positive for all differentiable real functions of a real variable. This example shows us that there is no guarantee that a given DE has even one solution.

Example 7. The DE $(dy/dx)^2+y^2=0$ has the *unique* solution f with specified domain $(-\infty,+\infty)$ where $y=f(x)\equiv 0$; that is, the DE has *one and only one* solution. If a solution g existed on $(-\infty,+\infty)$ where $g(x_0)\neq 0$ at some $x=x_0$, the left member of the DE would be positive for $x=x_0$. Note that the solution of this DE involves no arbitrary constants.

Example 8. Find solutions of $y'=x^{-1}$ on $(-\infty,0)$.

Solution: Since $d/dx(\ln|x|+C)=x^{-1}$ on $(-\infty,0)$, solutions are given by $y=\ln|x|+C$. Since $|x|=-x$ on $(-\infty,0)$, the solutions are also given by $y=\ln(-x)+C$. The DE has an infinite number of solutions, each having the specified domain $(-\infty,0)$. By $\ln u$ we mean the natural, or Napierian, logarithm of u.

Definition The order of a differential equation is the order of the highest-order derivative appearing in the equation.

The DE of Examples 1–8 are first-order equations.

Example 9. The DE $y''+xy'=0$ is a second-order equation.

Example 10. The DE $y'''-x^2=0$ is a third-order equation.

Definition An ordinary DE is said to be *linear* if and only if it can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

where f and the coefficients $a_0, a_1, \ldots, a_n$ are continuous functions of x. All other DE are called *nonlinear*. (NOTE: $y^{(n)}$ denotes d^ny/dx^n.)

Thus a linear DE is linear in y and its derivatives.

EXAMPLE 11. The DE $y'' + xy' + x^2y - e^x = 0$ is linear, since it is linear in y, y', and y''.

EXAMPLE 12. The DE $y'' + \cos y = 0$ is nonlinear, since it is not linear in y.

EXAMPLE 13. The DE $y'' + yy' + x = 0$ is nonlinear, since y, the coefficient of y', denotes an unknown function of x instead of a specific function of x.

In Chapter 5 we will encounter the linear DE

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{d}{dt}E(t)$$

and the nonlinear DE

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

In the first DE, which governs the flow of current in an electric circuit, L, R, and C are constants, i denotes the current in amperes, and E is the voltage at time t.

In the second DE, which governs the motion of a simple pendulum, g and l are constants and θ denotes the angle between the pendulum rod and the vertical.

The first-order DE $y' = \cos x$ has a solution given by $y = \sin x$, valid for all x. It also has an infinite number of solutions given by $y = \sin x + C$, where C is an arbitrary constant.

The second-order DE $y'' = 6x$ has solutions given by $y = x^3 + C_1x + C_2$, valid for all x, where C_1 and C_2 are arbitrary constants. These solutions are found by integrating $y'' = 6x$ to obtain $y' = 3x^2 + C_1$, and then integrating a second time.

In general, the solution of an nth-order DE contains n arbitrary constants. The n constants are said to be essential if it is not possible to write the solution in a form involving fewer than n constants. For example, the function given by Ae^{B+x} appears to contain two essential arbitrary constants, but in fact contains only one. This is readily seen by writing

$$Ae^{B+x} = Ae^B e^x = Ce^x$$

where Ae^B is replaced by the single arbitrary constant C. (For a more complete discussion of essential arbitrary constants see Reference 1.1.) We shall assume that any constants appearing in a solution are essential unless we state otherwise.

Definition By a *general solution* of an nth-order DE, we mean a solution containing n essential arbitrary constants.

If *every* solution of a DE can be obtained by assigning particular values to the n arbitrary constants in a general solution, that general solution is called a *complete solution*. A solution of the DE that cannot be obtained from a general solution by assigning particular values to the arbitrary constants is called a *singular solution*.

The term *general solution* is unsatisfactory since a general solution may or may not be a complete solution. Some authors use the term "general solution" only when treating linear DE. The reason for this will appear in Chapter 4. More theoretical developments prefer to focus attention on differential systems that include DE and to prove existence and uniqueness theorems for such systems. (This approach will be discussed in Sections 1.2 and 2.7.)

Example 14. The solution given by $y = \sin x + C$ is a general solution of the DE $y' = \cos x$. This solution is also a complete solution, since two functions having a derivative given by $\cos x$ can differ by at most a constant.

Example 15. The solution given by $y = Cx - C^2$, where C is an arbitrary constant, is a general solution of the first-order DE $(dy/dx)^2 - x\,dy/dx + y = 0$. This is not a complete solution, since the DE also has the singular solution given by $y = x^2/4$. The singular solution cannot be obtained from $y = Cx - C^2$ by assigning a particular value to C.

Definition Any solution of a DE that can be obtained from a general solution by assigning values to the essential arbitrary constants is called a *particular solution*.

For example, by setting $C = 0$ in Example 14, we obtain the particular solution given by $y = \sin x$ of the DE $y' = \cos x$.

Example 16. Solutions of the DE $y' = 2x^{-3}$ are given by $y = C - x^{-2}$, and the domain of each solution is the set $S = (-\infty, 0) \cup (0, +\infty)$.

The following example illustrates a partial DE. (We shall consider partial DE in Chapter 10.)

EXAMPLE 17. Show that the function f defined by $z = f(x, t) = (x - 4t)^2$ is a solution of the *partial* DE

$$\frac{\partial^2 z}{\partial t^2} = 16\,\frac{\partial^2 z}{\partial x^2}$$

Solution: From

$$\frac{\partial z}{\partial x} = 2(x - 4t), \quad \frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial t} = -8(x - 4t), \quad \frac{\partial^2 z}{\partial t^2} = 32$$

we obtain $32 = 16(2)$, true for all x and all t. The domain of f is the set of all ordered pairs (x, t) of real numbers.

A special type of DE is classified according to degree.

Definition If the DE

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.2}$$

can be expressed as a polynomial in $y, y', y'', \ldots, y^{(n)}$, the exponent of the highest-order derivative is called the *degree* of the DE.

EXAMPLES. The differential equations

$$y''' - x^2 = 0 \qquad y'' - y'^2 + x = 0 \qquad y'' - 3y' + 2y - e^x = 0$$

$$x^2y'' + xy' + x^2 = 0 \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

are first-degree equations. The DE

$$(y')^2 - xy' + y = 0 \quad \text{and} \quad (y')^2 - xy^3 = 0$$

are second-degree equations. The DE $y'' = \pm\sqrt{1 + y'}$ is also a second-degree equation since it can be written in the form $(y'')^2 - y' - 1 = 0$. The DE $y'' - \ln y = 0$ and $y' - \cos y = 0$ have no degree since neither can be written in the form (1.2).

We make the usual assumption that the DE

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.2}$$

can be solved for the highest-order derivative appearing; that is, that it can be