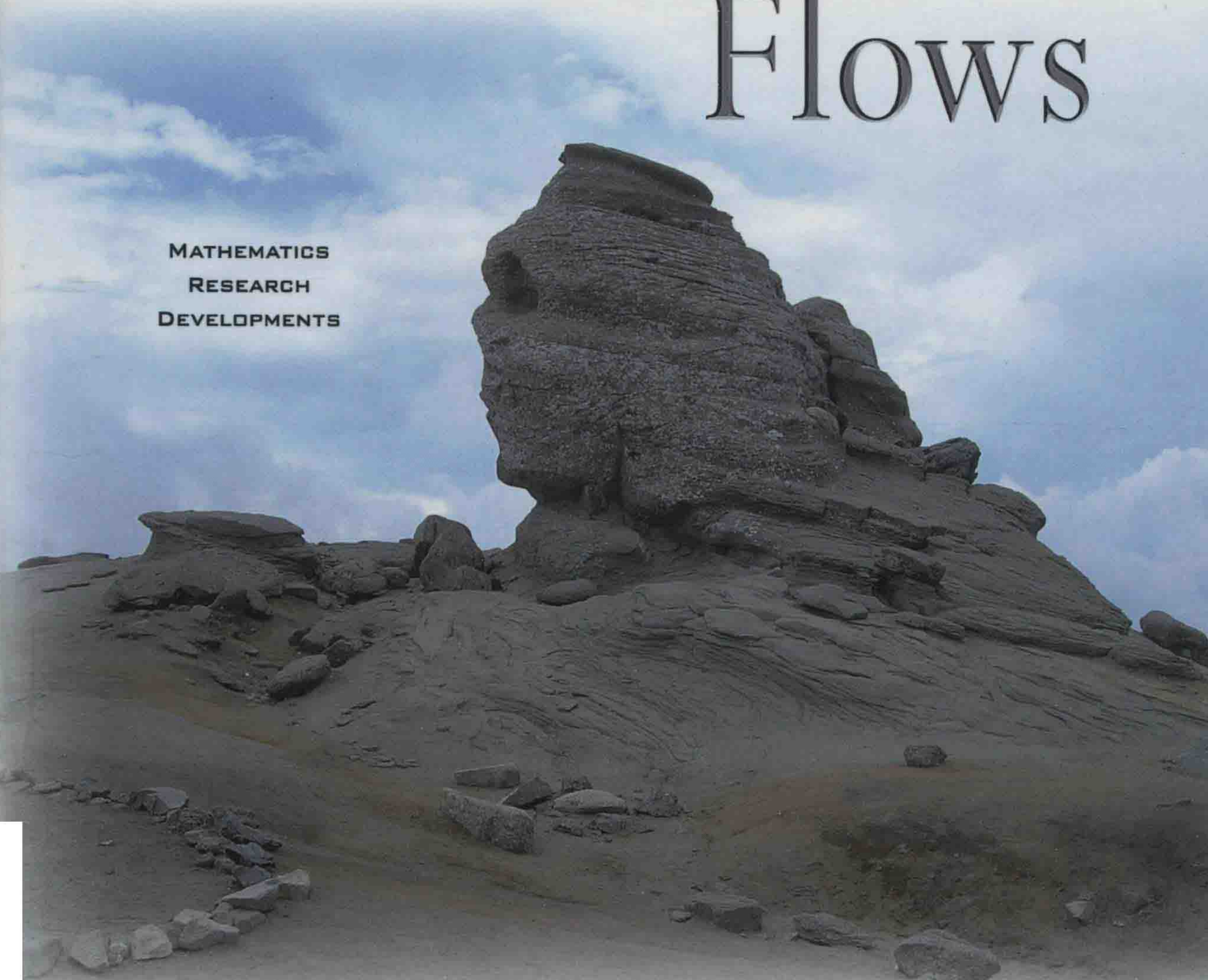
Binary Periodic Signals and Flows



MATHEMATICS RESEARCH DEVELOPMENTS

SERBAN E. VLAD

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BINARY PERIODIC SIGNALS AND FLOWS

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Preface

The Boolean autonomous deterministic regular asynchronous systems have been defined for the first time in our work [16] and a deeper study of such systems can be found in [18]. The concept has its origin in switching theory, the theory of modeling the switching circuits from the digital electrical engineering. The attribute Boolean vaguely refers to the Boole algebra with two elements; autonomous means that there is no input; determinism means the existence of a unique state function; and regular indicates the existence of a function $\Phi : \{0,1\}^n \to \{0,1\}^n, \Phi = (\Phi_1, ..., \Phi_n)$ that 'generates' the system. The time set is discrete: $\{-1, 0, 1, ...\}$ or continuous: **R**. The system, which is analogue to the (real, usual) dynamical systems, iterates (asynchronously) on each coordinate $i \in \{1, ..., n\}$, one of

- Φ_i : we say that Φ is computed, at that time instant, on that coordinate;

- $\{0,1\}^n \ni (\mu_1, ..., \mu_i, ..., \mu_n) \longmapsto \mu_i \in \{0,1\}$: we use to say that Φ is not computed, at that time instant, on that coordinate.

The flows are these that result by analogy with the dynamical systems.

The 'nice' discrete time and real time functions that the (Boolean) asynchronous systems work with are called signals and periodicity is a very important feature in Nature.

In the first two chapters we give the most important concepts concerning the signals and periodicity. The periodicity properties are used to characterize the eventually constant signals in Chapter 3 and the constant signals in Chapter 4. Chapters 5,...,8 are dedicated to the eventually periodic points, eventually periodic signals, periodic points and periodic signals.

Chapter 9 shows constructions that, given an (eventually) periodic point, by changing some values of the signal, change the periodicity properties of the point.

The monograph continues with flows. Chapter 10 is dedicated to the computation functions, i.e. to the functions that show when and how the function Φ is iterated (asynchronously). Chapter 11 introduces the flows and Chapter 12 gives a wider point of view on the flows, which are interpreted as deterministic asynchronous systems. Chapters 13,...,15 restate the topics from Chapters 3,...,8 in the special case when the signals are flows and the main interest is periodicity.

The bibliography consists in general in works of (real, usual) dynamical systems and we use analogies.

The book ends with a list of notations, an index of notions and an appendix with lemmas. These lemmas are frequently used in the exposure and some of them are interesting by themselves.

The book is structured in chapters, each chapter consists in several sections and each section is structured in paragraphs. The chapters begin with an abstract. The paragraphs

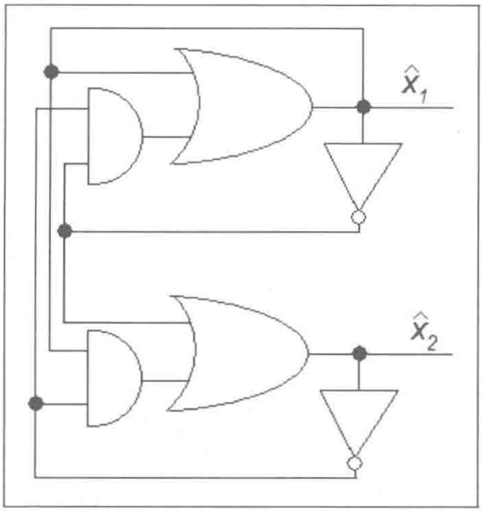

Figure 1. Asynchronous circuit.

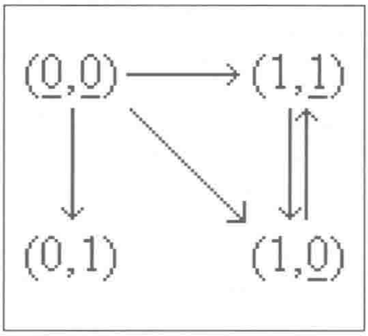

Figure 2. The state diagram of the circuit from Figure 1.

are of the following kinds: definitions, notations, remarks, theorems, corollaries, lemmas, examples and propositions. Each kind of paragraph is numbered separately on the others. Inside the paragraphs, the equations and, more generally, the most important statements are numbered also. When we refer to the statement (x,y) this means the $y-th$ statement of the $x-th$ section of the current chapter. Sometimes we write $(x,y)_{page\ z}$ in order to indicate the page where the statement occurs.

We refer to a definition, theorem, example,... by indicating its number and, when necessary, its page.

In order to point out our source of inspiration, we give the example of the circuit from Figure 1, where $\widehat{x}:\{-1,0,1,...\}\longrightarrow\{0,1\}^2$ is the signal representing the state of the system, and the initial state is $\widehat{x}(-1)=(0,0)$. The function that generates the system is $\Phi:\{0,1\}^2\longrightarrow\{0,1\}^2$, $\forall\mu\in\{0,1\}^2$,

$$\Phi(\mu)=(\mu_1\cup\overline{\mu_1}\cdot\overline{\mu_2},\overline{\mu_1}\cup\mu_1\cdot\overline{\mu_2}).$$

The evolution of the system is given by its state portrait from Figure 2, where the arrows indicate the increase of time and we have underlined these coordinates $\mu_i, i=\overline{1,2}$ that, by the computation of Φ, change their value: $\Phi_i(\mu) = \overline{\mu_i}$. Let $\alpha : \{0,1,2,...\} \longrightarrow \{0,1\}^2$ be the computation function whose values α_i^k show that Φ_i is computed at the time instant k if $\alpha_i^k = 1$, respectively that it is not computed at the time instant k if $\alpha_i^k = 0$, where $i = \overline{1,2}$ and $k \in \{0,1,2,...\}$. The uncertainty related with the circuit, depending in general on the technology, the temperature, etc. manifests in the fact that the order and the time of computation of each coordinate function Φ_i are not known. If the second coordinate is computed at the time instant 0, then $\alpha^0 = (0,1)$ indicates the transfer from $(0,0)$ to $(0,1)$, where the system remains indefinitely long for any values of $\alpha^1, \alpha^2, \alpha^3, ...$, since $\Phi(0,1) = (0,1)$. Such a signal $\widehat{x}$ is called eventually constant and it corresponds to a stable system. The eventually constant discrete time signals are eventually periodic with an arbitrary period $p \geq 1$.

Another possibility is that the first coordinate of Φ is computed at the time instant 0, thus $\alpha^0 = (1,0)$. Figure 2 indicates the transfer from $(0,0)$ to $(1,0)$, while $\alpha^0 = (1,1)$ indicates the transfer from $(0,0)$ to $(1,1)$, as resulted by the simultaneous computation of $\Phi_1(0,0)$ and $\Phi_2(0,0)$. And if $\alpha^k = (1,1), k \in \{1,2,...\}$, then $\widehat{x}$ is eventually periodic with the period $p \in \{2,4,6,...\}$, as it switches from $(1,1)$ to $(1,0)$ and from $(1,0)$ to $(1,1)$. This last possibility represents an unstable system.

An interesting open problem that we have reached during these studies (mentioned in Chapter 6) is the following one[1]. We suppose that time is real $t \in \mathbf{R}$. If a signal x is periodic, then its points[2] $x(t)$ are periodic

$$\begin{cases} \exists T > 0, \forall t \in \mathbf{R}, \forall z \in \mathbf{Z}, x(t) = x(t+zT) \\ \Longrightarrow \forall t \in \mathbf{R}, \exists T > 0, \forall z \in \mathbf{Z}, x(t) = x(t+zT), \end{cases}$$

and the set of periods of the signal is the intersection of the sets of periods of its points. Is the inverse statement true?

$$\begin{cases} \forall t \in \mathbf{R}, \exists T > 0, \forall z \in \mathbf{Z}, x(t) = x(t+zT) \\ \overset{?}{\Longrightarrow} \exists T > 0, \forall t \in \mathbf{R}, \forall z \in \mathbf{Z}, x(t) = x(t+zT) \end{cases}$$

In other words, if all the points $x(t)$ are periodic, the possibility exists that the intersection of the sets of their periods is empty? If so, the signal is not periodic.

The book addresses to researchers in systems theory and computer science, but it is also interesting to those that study periodicity itself. From this last perspective, the binary signals may be thought of as functions with finitely many values.

The author is aware of the fact that the exposure could have been occasionally shorter, or simpler, or perhaps more correct. For this reason, he thanks in advance to those that would accept to make suggestions of improvements for the next editions of the book.

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Oradea, December 2015

[1]Because we want to make ourselves understood, we state this open problem under a more general form than the needs of the exposure of the monograph.

[2]point of x = value $x(t)$ of x in some t.

Contents

Chapter 1

Preliminaries

The signals from digital electrical engineering are modeled by 'nice' discrete time and real time functions, which are also called signals and their introduction is the purpose of this chapter. We define the left and the right limits of the real time signals, the initial and the final values of the signals, the initial and the final time of the signals, the forgetful function, the orbits, the omega limit sets and the support sets. The last section refers to the images of the signals via a function.

1. The Definition of the Signals

Notation 1. *We denote by* $\mathbf{B} = \{0, 1\}$ *the binary Boole algebra. Its laws are the usual ones:*

$\overline{}$			$\cdot$	0	1		$\cup$	0	1		$\oplus$	0	1
0	1	;	0	0	0	;	0	0	1	;	0	0	1
1	0		1	0	1		1	1	1		1	1	0

Table 1

called negation, not, or (logical) complement; product or intersection; sum or union; and modulo 2 sum or disjoint union. These laws induce laws that are denoted with the same symbols on $\mathbf{B}^n, n \geq 1$.

Definition 1. *Both sets* $\mathbf{B}$ *and* $\mathbf{B}^n$ *are organized as topological spaces by the discrete topology.*

Notation 2. $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ *denote the sets of the non negative integers, of the integers and of the real numbers.* $\mathbf{N}_{\_} = \mathbf{N} \cup \{-1\}$ *is the notation of the discrete time set.*

Notation 3. *We denote*

$$\widehat{Seq} = \{(k_j) | k_j \in \mathbf{N}_{\_}, j \in \mathbf{N}_{\_} \text{ and } k_{-1} < k_0 < k_1 < \ldots\},$$

$$Seq = \{(t_k) | t_k \in \mathbf{R}, k \in \mathbf{N} \text{ and } t_0 < t_1 < t_2 < \ldots \text{ superiorly unbounded}\}.$$

Example 1. *A typical example of element of* $\widehat{Seq}$ *is the sequence* $k_j = j, j \in \mathbf{N}_{\_}$ *and typical examples of elements of* Seq *are given by the sequences* $z, z+1, z+2, \ldots, z \in \mathbf{Z}$.

Proposition 1. *Let* $(t_k) \in Seq$ *and* $t \in \mathbf{R}$ *be arbitrary. Then*

$$\exists \varepsilon > 0, \{k | k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \begin{cases} \{k'\}, if\ t = t_{k'}, \\ \varnothing, if\ \forall k \in \mathbf{N}, t \neq t_k. \end{cases}$$

Proof. We have the following possibilities.

Case $t < t_0$; we take $\varepsilon \in (0, t_0 - t)$, for which $\{k|k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \varnothing$.

Case $t = t_0$; for $\varepsilon \in (0, t_1 - t)$ we have $\{k|k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \{0\}$.

Case $t \in (t_{k'-1}, t_{k'}), k' \geq 1$; $\varepsilon \in (0, \min\{t - t_{k'-1}, t_{k'} - t\})$ gives $\{k|k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \varnothing$.

Case $t = t_{k'}, k' \geq 1$; in this situation any $\varepsilon \in (0, \min\{t - t_{k'-1}, t_{k'+1} - t\})$ gives $\{k|k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \{k'\}$. □

Remark 1. *The previous* ε *obviously depends on* t. *On the other hand, let us consider the sequence* $t_k = \frac{1}{0+1} + \frac{1}{1+1} + ... + \frac{1}{k+1}, k \in \mathbf{N}$. *We notice that* $(t_k) \in Seq$ *and*

$$\forall \varepsilon > 0, \exists t \in \mathbf{R}, card(\{k|k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\}) > 1$$

holds.

Notation 4. $\chi_A : \mathbf{R} \to \mathbf{B}$ *is the notation of the characteristic function of the set* $A \subset \mathbf{R} : \forall t \in \mathbf{R}$,

$$\chi_A(t) = \begin{cases} 1, if\ t \in A, \\ 0, otherwise. \end{cases}$$

Definition 2. *The* ***discrete time signals*** *are by definition the functions* $\widehat{x} : \mathbf{N}_{\_} \to \mathbf{B}^n$. *Their set is denoted with* $\widehat{S}^{(n)}$.

The ***continuous time signals*** *are the functions* $x : \mathbf{R} \to \mathbf{B}^n$ *of the form* $\forall t \in \mathbf{R}$,

$$x(t) = \mu \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus ... \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus ... \tag{1.1}$$

where $\mu \in \mathbf{B}^n$ *and* $(t_k) \in Seq$. *Their set is denoted by* $S^{(n)}$.

Example 2. *The constant functions* $\widehat{x} \in \widehat{S}^{(1)}, x \in S^{(1)}$ *equal with* $\mu \in \mathbf{B}$:

$$\forall k \in \mathbf{N}_{\_}, \widehat{x}(k) = \mu, \tag{1.2}$$

$$\forall t \in \mathbf{R}, x(t) = \mu \tag{1.3}$$

are typical examples of signals. Here are some other examples:

$$\forall k \in \mathbf{N}_{\_}, \widehat{x}(k) = \begin{cases} 1, if\ k\ is\ odd, \\ 0,\ if\ k\ is\ even, \end{cases} \tag{1.4}$$

$$\forall t \in \mathbf{R}, x(t) = \chi_{[0,\infty)}(t), \tag{1.5}$$

$$\forall t \in \mathbf{R}, x(t) = \chi_{[0,1)}(t) \oplus \chi_{[2,3)}(t) \oplus ... \oplus \chi_{[2k,2k+1)}(t) \oplus ... \tag{1.6}$$

The signal from (1.5) is called the (unitary) step function (of Heaviside).

Remark 2. *At Definition 2 a convention of notation has occurred for the first time, namely a hat '$\widehat{\ }$' is used to show that we have discrete time. The hat will make the difference between, for example, the notation of the discrete time signals $\widehat{x}, \widehat{y}, ..$ and the notation of the real time signals $x, y, ...$*

Remark 3. *The discrete time signals are sequences. The real time signals are piecewise constant functions.*

Remark 4. *As we shall see in the rest of the book, the study of the periodicity of the signals does not use essentially the fact that they take values in $\mathbf{B}^n$, but the fact that they take finitely many values. For example, instead of using '$\cdot$' and '$\oplus$' in (1.1), we can write equivalently*

$$x(t) = \begin{cases} \mu, t < t_0, \\ x(t_0), t \in [t_0, t_1), \\ \dots \\ x(t_k), t \in [t_k, t_{k+1}), \\ \dots \end{cases}$$

Remark 5. *The signals model the electrical signals of the circuits from the digital electrical engineering.*

2. Left and Right Limits

Theorem 1. *For any $x \in S^{(n)}$ and any $t \in \mathbf{R}$, there exist $x(t-0), x(t+0) \in \mathbf{B}^n$ with the property*

$$\exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0), \tag{2.1}$$

$$\exists \varepsilon > 0, \forall \xi \in (t, t+\varepsilon), x(\xi) = x(t+0). \tag{2.2}$$

Proof. We presume that x, t are arbitrary and fixed and that x is of the form

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus \dots \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus \dots \tag{2.3}$$

with $\mu \in \mathbf{B}^n$ and $(t_k) \in Seq$. We take $\varepsilon > 0$ small enough, see Proposition 1, page 2 such that

$$\{k | k \in \mathbf{N}, t_k \in (t-\varepsilon, t+\varepsilon)\} = \begin{cases} \{k'\}, if\ t = t_{k'}, \\ \varnothing, if\ \forall k \in \mathbf{N}, t \neq t_k. \end{cases}$$

We have the following possibilities:

Case $t < t_0$;

$$\forall \xi \in (t-\varepsilon, t), x(\xi) = \mu,$$
$$\forall \xi \in (t, t+\varepsilon), x(\xi) = \mu.$$

Case $t = t_0$;

$$\forall \xi \in (t-\varepsilon, t), x(\xi) = \mu,$$
$$\forall \xi \in (t, t+\varepsilon), x(\xi) = x(t_0).$$

Case $t \in (t_{k'-1}, t_{k'}), k' \geq 1$;

$$\forall \xi \in (t - \varepsilon, t), x(\xi) = x(t_{k'-1}),$$

$$\forall \xi \in (t, t + \varepsilon), x(\xi) = x(t_{k'-1}).$$

Case $t = t_{k'}, k' \geq 1$;

$$\forall \xi \in (t - \varepsilon, t), x(\xi) = x(t_{k'-1}),$$

$$\forall \xi \in (t, t + \varepsilon), x(\xi) = x(t_{k'}).$$

□

Definition 3. *The functions $\mathbf{R} \ni t \to x(t-0) \in \mathbf{B}^n, \mathbf{R} \ni t \to x(t+0) \in \mathbf{B}^n$ are called the **left limit** function of x and the **right limit** function of x.*

Remark 6. *Theorem 1 states that the signals $x \in S^{(n)}$ have a left limit function $x(t-0)$ and a right limit function $x(t+0)$. Moreover, if (2.3) is true, then*

$$x(t-0) = \mu \cdot \chi_{(-\infty,t_0]}(t) \oplus x(t_0) \cdot \chi_{(t_0,t_1]}(t) \oplus \ldots \oplus x(t_k) \cdot \chi_{(t_k,t_{k+1}]}(t) \oplus \ldots, \tag{2.4}$$

$$x(t+0) = x(t) \tag{2.5}$$

hold, meaning in particular that $x(t-0)$ is not a signal and that $x(t+0)$ coincides with $x(t)$.

Remark 7. *The property (2.5) stating in fact that the real time signals x are right continuous will be used later under the form*

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in [t, t+\varepsilon), x(\xi) = x(t). \tag{2.6}$$

3. Initial and Final Values, Initial and Final Time

Definition 4. *The **initial value** of $\widehat{x} \in \widehat{S}^{(n)}$ is $\widehat{x}(-1) \in \mathbf{B}^n$.*

For $x \in S^{(n)}$,

$$x(t) = \mu \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus \ldots \oplus x(t_k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus \ldots, \tag{3.1}$$

*where $\mu \in \mathbf{B}^n$ and $(t_k) \in Seq$, the **initial value** is μ.*

Notation 5. *There is no special notation for the initial value of $\widehat{x}$.*

The initial value of x has two usual notations, $x(-\infty+0)$ and $\lim\limits_{t \to -\infty} x(t)$.

Definition 5. *By definition, the **initial time (instant)** of $\widehat{x}$ is $k = -1$.*

*The **initial time (instant)** of x is any number $t_0 \in \mathbf{R}$ that fulfills*

$$\forall t \leq t_0, x(t) = x(-\infty + 0). \tag{3.2}$$

Notation 6. *The set of the initial time instants of x is denoted by I^x.*