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LINEAR AND QUASI-LINEAR EQUATIONS OF PARABOLIC TYPE

BY

O. A. LADYŽENSKAJA
V. A. SOLONNIKOV
N. N. URAL'CEVA

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OF
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O. A. LADYŽENSKAJA
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ПАРАБОЛИЧЕСКОГО ТИПА

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PREFACE

Equations of parabolic type are encountered in many branches of mathematics and mathematical physics, and the forms in which they are investigated vary widely. The equations encountered most frequently (and in adjoining fields of study almost exclusively) are those of second order. Such equations (and certain classes of systems of second order), both linear and quasi-linear, make up the subject of investigation of the present book. Our study of these equations is concerned mainly with the solvability of their boundary value problems and with an analysis of the connections between the smoothness of the solutions and the smoothness of the known functions entering into the problem.

A basic condition that is assumed to be fulfilled for all equations considered is the condition of uniform parabolicity. For such equations we have managed to give sufficiently complete answers to central questions on the solvability of the above-indicated problems and to establish a series of exact dependences of the properties of the solutions on the properties of the known functions in terms of their mutual membership in the most commonly occurring function spaces.

For linear equations the solvability of the basic boundary value problems and of the Cauchy problem depends only on the smoothness of the functions defining the problem (i. e. the functions considered to be known in the problem, namely the coefficients and the free terms of the equations, the functions assigning the initial and boundary conditions and the boundary of the domain in which the solution exists). The smoother these known functions, the better behaved will be the solution. Conversely, if one worsens the properties of the known functions in the problem, then the differential properties of the solutions also become worse, where the deterioration (as would equally be true with an improvement) has a local character (for example, the smoothness of the solutions inside their domain of definition is determined only by the smoothness of the coefficients and free terms of the equation and does not depend on the smoothness of the boundary or of the initial and boundary functions). But one cannot arbitrarily worsen the properties of the functions defining the problem (for example, admit in the coefficients singularities of high order). There exists a limit to admissible deteriorations, beyond which such properties of the problems as uniqueness are lost. As in the analysis carried out by us for elliptic equations in the book [65 q] we begin by determining this limit, for which we construct appropriate examples. With these examples (and examples from [65 l, n, o]) we have managed to outline with sufficient accuracy the limits of a possible theory of boundary value problems for equations with discontinuous and, in general, unbounded coefficients and free terms, which is later presented in Chapter III.

As a characteristic of such "bad" known functions we have selected their membership in the spaces $L_{q,r}(Q_T)$.[1] The solutions here fall into a certain function space, the elements of which have derivatives of first order with respect to x and of order $\frac{1}{2}$ with respect to t. We then observe that the properties of these solutions improve as the differential properties of the functions defining the equation or problem improve.

A qualitatively different situation holds for nonlinear equations. For them the smoothness of the solutions and the solvability "in the large" of the boundary value problems and of the Cauchy problem is determined not only by the smoothness of the known functions $a_{ij}(x, t, u, p)$, $a(x, t, u, p)$ making up the equation but also by their behavior as u and p increase without limit. In §3 of Chapter I we cite a number of examples elucidating certain restrictions on this behavior, the nonfulfilment of which implies a nonsolvability of these problems "in the large." And in subsequent chapters (Chapters V, VI, VII) it is proved that these restrictions, together with a certain not large smoothness, are on the whole also sufficient for the unique solvability of the basic boundary value problems and of the Cauchy problem for quasi-linear equations.

The general plan of the book is as follows. In Chapter I we present the basic notation and terminology used in the book, a description of the main results proved in it, and a number of examples indicating the exactness of these results; finally, we give a brief historical survey. In Chapter II we have assembled propositions that are used throughout the book and describe the properties, not of the solutions of any differential equations, but of arbitrary functions belonging to various function spaces or classes. It is perhaps better to treat this chapter as a reference on

---

[1] For functions $u(x, t)$ of a space $L_{q,r}(Q_T)$ the norm

$$\|u\|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |u|^q \, dx \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}}.$$

is finite.

its different assertions. The main text begins with Chapter III. It and Chapter IV are devoted to linear equations. In Chapters V and VI we investigate quasi-linear equations. Finally, in Chapter VII we examine linear and quasi-linear systems of second order with common principal parts and give a survey of the results on general boundary-value problems for linear parabolic systems, the most general of those considered up to the present time. The main contents of each chapter can be understood independently of the others.

The contents of all the chapters, except Chapter IV and parts of Chapters II and VII, are based on the work of O. A. Ladyženskaja and N. N. Ural'ceva. These chapters were written by them. Chapter IV and §§ 8–10 of Chapter VII were written by V. A. Solonnikov, who is responsible for many of the results in this part of the book.

The authors are extremely grateful to Academician V. I. Smirnov for having looked over the manuscript of the entire book and having made a number of important critical remarks and suggestions. They were taken into account during the final revision.

The authors express their heartfelt thanks to their colleagues and students A. Treskunov, A. Oskolkov, M. Faddeev, I. Krol', V. Matveev and technician L. M. Dikušina for their help in the preparation of the book. A particularly large amount of quite expert assistance was rendered by A. Treskunov, a graduate student at Leningrad University, who worked with us throughout the writing of the book and obtained during this time some interesting results on linear equations (see Bibliography).

Prefatory Note to the Translation

The active cooperation of the Russian authors has made it possible to bring the present translation up-to-date and to improve it in several respects. Slight additions and corrections have been made throughout, and some of the material has been entirely rewritten, most notably Chapter II §2 on embedding theorems, Chapter IV §4 on certain supplementary theorems, and Chapter V §6 on solvability of the first boundary problem. The translator and the editorial staff wish to thank the Russian authors for their long-continued and cordial assistance.

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CHAPTER I

INTRODUCTORY MATERIAL

This book is devoted to the basic linear and quasi-linear second-order partial differential equations of parabolic type. For them the solvability of the basic boundary value problems and of the Cauchy problem in various function spaces is studied and investigations are carried out concerning the dependences of the smoothness properties of the solutions of these equations on the known functions making up the equations and on the properties of the other known functions in the problems. We begin with a description of certain examples that permit one to outline with sufficient accuracy the contours of a possible theory for these questions, and with an enumeration of the basic results of the present book. These sections (§§ 3 and 4) may be usefully reread after making an acquaintance with Chapters III and IV. Preceding them (§2) is a description of the statements of the basic problems for parabolic equations and an account of one of the basic properties inherent in the solutions of parabolic equations of second order namely, in their classical form, the maximum principle. Later on, both the statements of the problems and the maximum principle acquire a different form, appropriate to the function space containing the solutions being investigated. These modifications in the classical form and the methods created for working with them led to success in studying quasi-linear equations "in the large" and linear equations with bad coefficients.

Although it can be read independently, the present book has much in common, in regard to the methods of investigation, with the book by O. A. Ladyženskaja and N. N. Ural'ceva, "Linear and quasi-linear equations of elliptic type" [65q].

All functions, arguments and parameters considered in this book are real. An exception occurs with §§.4 and 18 of Chapter III and §8 and 9 of Chapter VII, where Fourier and Laplace transforms are used.

§1. BASIC NOTATION AND TERMINOLOGY

1. Abridged notation.

E_n is the n-dimensional euclidean space; $x = (x_1, \cdots, x_n)$ is an arbitrary point in it.

E_{n+1} is the $(n+1)$-dimensional euclidean space; its points are denoted by (x, t), where x is in E_n and t is in $(-\infty, \infty)$.

Ω is a domain in E_n, i.e. an arbitrary open connected set of points of E_n. In all chapters except IV, unless otherwise stated, Ω is considered to be a bounded domain. In Chapter IV the letter Ω denotes an arbitrary domain.

S is the boundary of Ω.

$\bar{\Omega}$ is the closure of Ω, so that $\bar{\Omega} = \Omega \cup S$.

K_ρ is an arbitrary (open) ball in E_n of radius ρ, $\kappa_n = \operatorname{mes} K_1$, and ω_n is the surface area of K_1.

$\Omega_\rho = K_\rho \cap \Omega$.

Q_T is the cylinder $\Omega \times (0, T)$, i.e. the set of points (x, t) of E_{n+1} with $x \in \Omega$, $t \in (0, T)$.

Q' is an arbitrary open subset of Q_T.

S_T is the lateral surface of Q_T, or more precisely the set of points (x, t) of E_{n+1} with $x \in S$, $t \in [0, T]$.

$\Gamma_T = S_T \cup \{(x, t)\colon x \in \Omega,\ t = 0\}$.

$S_0 = \{(x, t)\colon x \in S,\ t = 0\}$; $\Gamma_0 = \{(x, t)\colon x \in \bar{\Omega},\ t = 0\}$.

$Q_{t_1, t_2} = \Omega \times (t_1 < t < t_2)$.

$Q(\rho, \tau)$ is an arbitrary cylinder of the form $\{(x, t)\colon |x - x^0| < \rho;\ t_0 < t < t_0 + \tau\}$.

$Q(k, \rho, \tau)$ is the set of all points $(x, t) \in Q(\rho, \tau)$ at which the investigated function $u(x, t) > k$.

$Q_{t_1}(k)$ is the set of all points $Q_{t_1} = \Omega \times (0 < t < t_1)$, at which $u(x, t) > k$.

$\nu, \mu, \epsilon, \delta, \delta_k, \theta, \theta_k, \gamma, \alpha, \beta$ are positive constants, with α being assumed to belong to the interval $(0, 1)$.

$\nu(t)$ is a positive nonincreasing continuous function defined for $t \geq 0$.

$\mu(t)$ is a positive nondecreasing continuous function defined for $t \geq 0$.

δ_i^j is the Kronecker delta symbol: $\delta_i^i = 1$, $\delta_i^j = 0$ for $i \neq j$.

$$|x|=\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, \quad x^2=|x|^2,$$

$$p=(p_1, \ldots, p_n), \quad u_x=(u_{x_1}, \ldots, u_{x_n}), \quad |p|=\left(\sum_{i=1}^{n} p_i^2\right)^{\frac{1}{2}},$$

$$p^2=|p|^2, \quad |u_x|=\left(\sum_{i=1}^{n} u_{x_i}^2\right)^{\frac{1}{2}},$$

$$u_x^2=|u_x|^2, \quad u_{x_i}^2=(u_{x_i})^2,$$

$$|u_{xx}|=\left(\sum_{i,\,j=1}^{n} u_{x_ix_j}^2\right)^{\frac{1}{2}},$$

$$a(x, t, u, p)=a(x_1, \ldots, x_n, t, u, p_1, \ldots, p_n),$$

$$a(x, t, u, u_x)=a(x_1, \ldots, x_n, t, u, u_{x_1}, \ldots, u_{x_n}).$$

ocs $\{u(x); \Omega\}$ is the oscillation of $u(x)$ on Ω, i.e. the difference between vrai $\max_{\Omega} u(x)$ and vrai $\min_{\Omega} u(x)$. osc $\{u(x, t); Q_T\}$ is defined analogously.

In the equations below we will encounter such expressions as

$$\frac{d}{dx_i}[a(x, t, u(x, t), u_x(x, t))],$$

which mean that in calculating the derivative d/dx_i it is necessary to take into account the presence of x_i not only in the first group of arguments but also in the other two, i.e. in the functions $u(x, t)$ and $u_x(x, t)$, so that

$$\frac{d}{dx_i}[a(x, t, u(x, t), u_x(x, t))]=\frac{\partial a}{\partial x_i}+\frac{\partial a}{\partial u}u_{x_i}+\frac{\partial a}{\partial u_{x_k}}u_{x_kx_i}.$$

Here and elsewhere pairs of equal indices imply a summation from 1 to n; in particular,

$$\frac{\partial a}{\partial u_{x_k}}u_{x_kx_i}=\sum_{k=1}^{n}\frac{\partial a}{\partial u_{x_k}}u_{x_kx_i}.$$

Sometimes, when it does not cause confusion, the symbol for total differentiation d/dx_i will be replaced by the more widely used symbol $\partial/\partial x_i$. For example, in a linear equation we usually write a term such as $(d/dx_i)(a_{ij}(x, t)u_{x_j}(x, t))$ in the form $(\partial/\partial x_i)(a_{ij}(x, t)u_{x_j}(x, t))$, even though in differentiating here one must take account of x_i in both $a_{ij}(x, t)$ and $u_{x_j}(x, t)$.

n is the outward (from Ω) unit normal to S at each of its points; $\partial/\partial \mathbf{n}$ denotes differentiation along **n**.

In Chapter VII the notation $\nu = -\mathfrak{n}$ is also used; ν is the inward unit normal to S.

A function $u(x)$ (or $u(x, t)$) is said to be *finite in* Ω (in Q_T) if it is different from zero only on some compact set that is separated from the boundary of Ω (of Q_T) by a positive distance.

A function $\zeta(x)$ (or $\zeta(x, t)$) is said to be *a cutting function for the domain* Ω (for Q_T) if it is continuous in $\bar{\Omega}$ (in $\bar{Q}_T$), has first-order piecewise-continuous bounded derivatives, vanishes on the boundary of this domain (on Γ_T), and has its values contained between zero and one.

2. Definitions of the basic function spaces. $L_q(\Omega)$ is the Banach space consisting of all measurable functions on Ω that are qth-power $(q \geq 1)$ summable on Ω. The norm in it is defined by the equalities

$$\|u\|_{q,\Omega} = \left(\int_\Omega |u(x)|^q \, dx\right)^{\frac{1}{q}} \quad \text{and} \quad \|u\|_{\infty,\Omega} = \underset{\Omega}{\operatorname{vrai\,max}} \, |u|.$$

Measurability and summability are to be understood everywhere in the sense of Lebesgue. The elements of $L_q(\Omega)$ are equivalence classes of the functions on Ω.

$L_{q,r}(Q_T)$ is the Banach space consisting of all measurable functions on Q_T with a finite norm

$$\|u\|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^q \, dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}},$$

where $q \geq 1$ and $r \geq 1$.

$L_{q,q}(Q_T)$ will be denoted by $L_q(Q_T)$, and the norm $\|\cdot\|_{q,q,Q_T}$ by $\|\cdot\|_{q,Q_T}$.

Generalized derivatives are to be understood in the way that is now customary in the majority of papers on differential equations. Different but equivalent definitions and their basic properties can be found, for example, in [112, Vol. V] and [113a].

$W_q^l(\Omega)$ for l integral is the Banach space consisting of all elements of $L_q(\Omega)$ having generalized derivatives of all forms up to order l inclusively, that are qth-power summable on Ω. The norm in $W_q^l(\Omega)$ is defined by the equality

$$\| u \|_{q,\Omega}^{(l)} = \sum_{j=0}^{l} \langle\langle u \rangle\rangle_{q,\Omega}^{(j)}, \tag{1.1}$$

where

$$\langle\langle u \rangle\rangle_{q,\Omega}^{(j)} = \sum_{(j)} \| D_x^j u \|_{q,\Omega}. \tag{1.2}$$

The symbol D_x^j denotes any derivative of $u(x)$ with respect to x of order j, while $\Sigma_{(j)}$ denotes summation over all possible derivatives of u of order j. For domains with "not too bad" boundaries $W_q^l(\Omega)$ coincides with the closure in the norm (1.1) of the set of all functions that are infinitely differentiable in $\bar{\Omega}$. This will be true, for example, for domains with piecewise-smooth boundaries (a definition of which is given below). Sometimes W_q^l is written in place of $W_q^l(\Omega)$, particularly if the domain Ω is subject to a further refinement.

$\overset{\bullet}{W}{}_q^l(\Omega)$ is the set of elements of $W_q^l(\Omega)$, that are finite in Ω.

$\overset{\circ}{W}{}_q^l(\Omega)$ is the subspace of $W_q^l(\Omega)$ in which the set of all functions that are infinitely differentiable and finite in Ω is dense. It is known that $\overset{\bullet}{W}{}_q^l(\Omega) \subset \overset{\circ}{W}{}_q^l(\Omega)$.

$W_q^{2l,l}(Q_T)$ for l integral $(q \geq 1)$ is the Banach space consisting of the elements of $L_q(Q_T)$ having generalized derivatives of the form $D_t^r D_x^s$ with any r and s satisfying the inequality $2r + s \leq 2l$. The norm in it is defined by the equality

$$\| u \|_{q,Q_T}^{(2l)} = \sum_{j=0}^{2l} \langle\langle u \rangle\rangle_{q,Q_T}^{(j)}, \tag{1.3}$$

where

$$\langle\langle u \rangle\rangle_{q,Q}^{(j)} = \sum_{(2r+s=j)} \| D_t^r D_x^s u \|_{q,Q_T}. \tag{1.4}$$

The summation $\Sigma_{(2r+s=j)}$ is taken over all nonnegative integers r and s satisfying the condition $2r + s = j$.

Spaces $W_q^l(\Omega)$ and $W_q^{l,l/2}(S_T)$ with nonintegral l will be used in Chapters IV and VII. The former space is defined in §2 of Chapter II, and the latter in §3 of Chapter II.

In addition to $W_q^{2l,l}(Q_T)$, we will encounter two spaces with different ratios of the upper indices:

$W_2^{1,0}(Q_T)$ is the Hilbert space with scalar product

$$(u, v)_{W_2^{1,0}(Q_T)} = \int_{Q_T} (uv + u_{x_k} v_{x_k})\,dx\,dt$$

and $W_2^{1,1}(Q_T)$ is the Hilbert space with scalar product

$$(u, v)_{W_2^{1,1}(Q_T)} = \int_{Q_T} (uv + u_{x_k} v_{x_k} + u_t v_t)\,dx\,dt.$$

$V_2(Q_T)$ is the Banach space consisting of all elements of $W_2^{1,0}(Q_T)$ having a finite norm

$$|u|_{Q_T} = \operatorname{vrai}\max_{0 \le t \le T} \|u(x, t)\|_{2,\Omega} + \|u_x\|_{2,Q_T}, \tag{1.5}$$

where here and below

$$\|u_x\|_{2,Q_T} = \sqrt{\int_{Q_T} u_x^2\,dx\,dt}.$$

$V_2^{1,0}(Q_T)$ is the Banach space consisting of all elements of $V_2(Q_T)$ that are continuous in t in the norm of $L_2(\Omega)$, with norm

$$|u|_{Q_T} = \max_{0 \le t \le T} \|u(x, t)\|_{2,\Omega} + \|u_x\|_{2,Q_T}. \tag{1.6}$$

The continuity in t of a function $u(x, t)$ in the norm of $L_2(\Omega)$ means that $\|u(x, t + \Delta t) - u(x, t)\|_{2,\Omega} \to 0$ for $\Delta t \to 0$. The space $V_2^{1,0}(Q_T)$ is obtained by completing the set $W_2^{1,1}(Q_T)$ in the norm of $V_2(Q_T)$.

$V_2^{1,1/2}(Q_T)$ is the subset of those elements $u(x, t)$ of $V_2^{1,0}(Q_T)$ for which

$$\int_0^{T-h} \int_\Omega h^{-1}\,[u(x, t+h) - u(x, t)]^2\,dx\,dt \xrightarrow[h \to 0]{} 0.$$

A zero over $W_2^{1,0}(Q_T)$, $W_2^{1,1}(Q_T)$, $V_2(Q_T)$, $V_2^{1,0}(Q_T)$, $V_2^{1,1/2}(Q_T)$ means that only those elements of these spaces are taken which vanish on S_T.

We now define spaces consisting of functions that are continuous in the sense of Hölder.

We will say that a function $u(x)$ defined in $\bar{\Omega}$ satisfies a Hölder condition in x with exponent α, $\alpha \in (0, 1)$, and Hölder constant $\langle u\rangle_{\Omega}^{(\alpha)}$ in the domain $\bar{\Omega}$ if

$$\sup \rho^{-\alpha} \operatorname{osc}\{u;\ \Omega_{\rho}^{i}\} \equiv \langle u\rangle_{\Omega}^{(\alpha)} < \infty, \tag{1.7}$$

where the supremum is taken over all connected components Ω_{ρ}^{i} of all Ω_{ρ} with $\rho \le \rho_0$. If the boundary of the domain Ω is "not too bad" (for example, piecewise-smooth without double points), then $\langle u\rangle_{\Omega}^{(\alpha)}$ can also be defined in another way, as

$$\sup_{\substack{x,\, x' \in \Omega \\ |x-x'| < \rho_0}} \frac{|u(x) - u(x')|}{|x - x'|^{\alpha}} = \langle u\rangle_{\Omega}^{(\alpha)}. \tag{1.8}$$

For domains with a two-piece boundary, for example for the domain $\{(x_1, x_2)\colon |x_1| < 1, |x_2| < 1 \text{ and } x_2 \neq 0 \text{ for } |x_1| \le \frac{1}{2}\}$, definitions (1.7) and (1.8) are not equivalent. In such cases we will adhere to the first definition.

We proceed to define the Hölder spaces $H^{l}(\bar{\Omega})$ and $H^{l,l/2}(\bar{Q}_T)$. In them l is always a nonintegral positive number.

$H^{l}(\bar{\Omega})$ is the Banach space whose elements are continuous functions $u(x)$ in Ω having in $\bar{\Omega}$ continuous derivatives up to order $[l]$ inclusively and a finite value for the quantity

$$|u|_{\Omega}^{(l)} = \langle u\rangle_{\Omega}^{(l)} + \sum_{j=0}^{[l]} \langle u\rangle_{\Omega}^{(j)}, \tag{1.9}$$

where

$$\langle u\rangle_{\Omega}^{(0)} = |u|_{\Omega}^{(0)} = \max_{\Omega} |u|,$$

$$\langle u\rangle_{\Omega}^{(j)} = \sum_{(j)} \left|D_x^{j} u\right|_{\Omega}^{(0)}, \quad \langle u\rangle_{\Omega}^{(l)} = \sum_{([l])} \langle D_x^{[l]} u\rangle_{\Omega}^{(l-[l])}$$

Equality (1.9) defines the norm $|u|_{\Omega}^{(l)}$ in $H^{l}(\bar{\Omega})$.

$H^{l,l/2}(\bar{Q}_T)$ is the Banach space of functions $u(x, t)$ that are continuous in $\bar{Q}_T$, together with all derivatives of the form $D_t^{r} D_x^{s}$ for $2r + s < l$, and have a finite norm

$$|u|_{Q_T}^{(l)} = \langle u\rangle_{Q_T}^{(l)} + \sum_{j=0}^{[l]} \langle u\rangle_{Q_T}^{(j)}, \tag{1.10}$$

where

$$\langle u\rangle_{Q_T}^{(0)} \equiv |u|_{Q_T}^{(0)} = \max_{Q_T} |u|,$$

$$\langle u\rangle_{Q_T}^{(j)} = \sum_{(2r+s=j)} \left|D_t^{r} D_x^{s} u\right|_{Q_T}^{(0)},$$

$$\langle u\rangle_{Q_T}^{(l)} = \langle u\rangle_{x,\, Q_T}^{(l)} + \langle u\rangle_{t,\, Q_T}^{(l/2)},$$

$$\langle u\rangle_{x,\, Q_T}^{(l)} = \sum_{(2r+s=[l])} \langle D_t^{r} D_x^{s} u\rangle_{x,\, Q_T}^{(l-[l])},$$