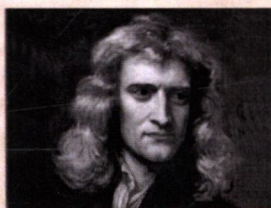


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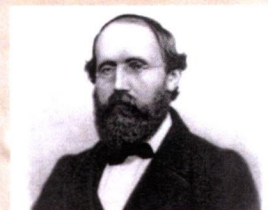
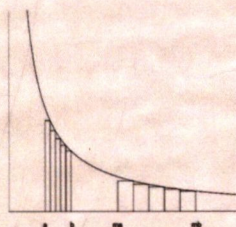
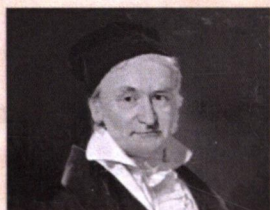


$$F(b) - F(a) = \int_a^b f(x) \, dx$$



$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} \, dx$$

$$e^{i\pi} = -1$$



Robert Carlson



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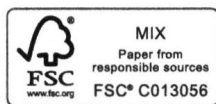
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Preface

This book is an introduction to real analysis, which might be briefly defined as the part of mathematics dealing with the theory of calculus and its more or less immediate extensions. Some of these extensions include infinite series, differential equations, and numerical analysis. This brief description is accurate, but somewhat misleading, since analysis is a huge subject which has been developing for more than three hundred years, and has deep connections with many subjects beyond mathematics, including physics, chemistry, biology, engineering, computer science, and even business and some of the social sciences.

The development of analytic (or coordinate) geometry and then calculus in the seventeenth century launched a revolution in science and world view. Within one or two lifetimes scientists developed successful mathematical descriptions of motion, gravitation, and the reaction of objects to various forces. The orbits of planets and comets could be predicted, tides explained, artillery shell trajectories optimized. Subsequent developments built on this foundation include the quantitative descriptions of fluid motion and heat flow. The ability to give many new and interesting quantitatively accurate predictions seems to have altered the way people conceived the world. What could be predicted might well be controlled.

During this initial period of somewhat over one hundred years, the foundations of calculus were understood on a largely intuitive basis. This seemed adequate for handling the physical problems of the day, and the very successes of the theory provided a substantial justification for the procedures. The situation changed considerably in the beginning of the nineteenth century. Two landmark events were the systematic use of infinite series of sines and cosines by Fourier in his analysis of heat flow, and the use of complex numbers and complex valued functions of a complex variable. Despite their ability to make powerful and accurate predictions of physical phenomenon, these tools were difficult to understand intuitively. Particularly in the area of Fourier series, some nonsensical results resulted from reasonable operations. The resolution of these problems took decades of effort, and involved a careful reexamination of the foundations of calculus. The ancient Greek treatment of geometry, with its explicit axioms, careful definitions, and emphasis on proof as a reliable foundation for reasoning, was used successfully as a model for the development of analysis.

A modern course in analysis usually presents the material in an efficient but austere manner. The student is plunged into a new mathematical environment,

replete with definitions, axioms, powerful abstractions, and an overriding emphasis on formal proof. Those students able to find their way in these new surroundings are rewarded with greatly increased sophistication, particularly in their ability to reason effectively about mathematics and its applications to such fields as physics, engineering and scientific computation. Unfortunately, the standard approach often produces large numbers of casualties, students with a solid aptitude for mathematics who are discouraged by the difficulties, or who emerge with only a vague impression of a theoretical treatment whose importance is accepted as a matter of faith.

This text is intended to remedy some of the drawbacks in the treatment of analysis, while providing the necessary transition from a view of mathematics focused on calculations to a view of mathematics where proofs have the central position. Our goal is to provide students with a basic understanding of analysis as they might need it to solve typical problems of science or engineering, or to explain calculus to a high school class. The treatment is designed to be rewarding for the many students who will never take another class in analysis, while also providing a solid foundation for those students who will continue in the "standard" analysis sequence.

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