FRACTIONAL KINETICS IN SPACE

Anomalous Transport Models

Vladimir Uchaikin • Renat Sibatov

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This book is first of its kind describing a new direction in modeling processes taking place in interplanetary and interstellar space (magnetic fields, plasma, cosmic rays, etc.). This method is based on a special mathematical analysis — fractional calculus. The reader will find in this book clear physical explanation of the fractional approach and will become familiar with basic rules in this calculus and main results obtained in frame of this approach. In spite of its profound subject, the book is not overloaded by mathematical details. It contains many illustrations, rich citation and remains accessible to a wide circle of physicists.

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"The Cosmos is all that is or was or ever will be. Our feeblest contemplations of the Cosmos stir us – there is a tingling in the spine, a catch in the voice, a faint sensation, as if a distant memory, of falling from a height. We know we are approaching the greatest of mysteries."

Carl Sagan

Contents

Overview

It is amazing how wildly different-sized objects exist in the Universe! The red giant Betelgeuse is so monstrously larger than our Sun that it could fill the orbit of Mars. But compared to our home Galaxy, the Milky Way, Betelgeuse is just a grain of sand on a beach. Clearly, these sizes are relative, and to distribute various objects over their sizes, we use a special term *size scale*. The size scale of an object is a region large enough to include the entire object but not so large that the object becomes insignificant.

Hermes Trismegistus, the founder of the philosophical school of Hermeticism, is famous for saying: ***"Know then the Greatest Secret of the Universe: As Above, So Below As Within, So Without"***. Nowadays, this idea is expressed by the term *scaling*. Scaling (or *self-similarity*, being a synonym of this term) of a dynamical process $x(t)$ is a special kind of its symmetry such that a change in scale of some variables can be compensated by a corresponding rescaling of others. Dealing with phenomena relating to different scales, we often meet the problems with graphical representation of dependencies under investigation for comparing them. To put the graphs with different scales on the same plot, the logarithmic transformation is often used

$$y = f(x) \quad \mapsto \quad \log y = \phi(\log x).$$

One of the specific properties of this transformation is its ability to straighten power function graphs:

$$y = x^\alpha \quad \mapsto \quad \log y = \alpha \log x.$$

This is an important sign of the property called the ***self-similarity***. We can change the units of both x and y, but the slope of the log-log plot remains the same. Functions of such kind characterize self-similar structures called ***fractals*** [Mandelbrot (1983)]. Many astrophysicists believe that interplanetary medium, interstellar medium, galaxy distributions possess fractal structure signs. The fractal concept lies in the base of modern description of turbulence in hydrodynamics and plasma.

The concept of self-similarity plays a leading role in the probability theory: you just recall the central limit theorem. Its extension to continuous time provides us such universal stochastic model as the *Brownian motion.* The first who saw the permanent chaotic motion of tiny pollen grains suspended in water was the Scotland botanist Robert Brown (1827). He was very surprised with this discovery and thought that this motion had the living origin. However, he would be much more surprised if he knew that his name will be associated with the movement of matter on giant cosmic scales! Namely this image had served as a basis for interpretation of many astrophysical phenomena and especially for description of cosmic rays propagation in the Galaxy [Berezinskii *et al.* (1984)].

Because of its chaotic character, Brownian motion does not look like Newtonian motion of planets along Keplerian orbits, and its mathematical description was not an easy task in those days. Only many decades later, Albert Einstein and independently of him Marian von Smoluchowski solved this problem on the base of the random process theory. The physical explanation of the phenomenon was based on assumption that atoms and molecules actually exist, and was later verified experimentally by Jean Perrin in 1908. Perrin was awarded the Nobel Prize in Physics in 1926 "for his work on the discontinuous structure of matter". But 19th centuries earlier Roman poet Titus Lucretius Carus ("On the Nature of Things") gave an instructive descripton of this phenomenon: "***You will see a multitude of tiny particles mingling in a multitude of ways... their dancing is an actual indication of underlying movements of matter that are hidden from our sight... It originates with the atoms which move of themselves.***"

In one-dimensional case, the Brownian *propagator* (probability density function for one tracer with a fixed initial point) has the self-similar form

$$p(x,t) = t^{-1/2} g(xt^{-1/2}), \quad -\infty < x < \infty,$$

where $g(x)$ is the Gaussian density. Brownian trajectories are continuous but nowhere differentiable curves (Fig. 0.1). The length of its segment between any two points of such curve is infinite, which is a sign of its fractality. Observe that giving t natural values 1,2,3,..., we bridge to the *random walk model* underlying such fundamental results of the probability theory as the ***Large Numbers Law*** and the ***Central Limit Theorem.***

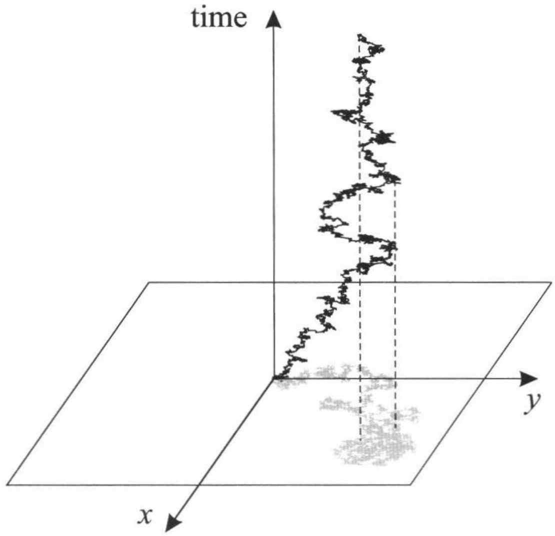

Fig. 0.1 A sample trajectory of the Brownian motion.

According to the latter, the size of a single walker probability cloud after N steps is

$$R(N) = R(1)N^{1/2}.$$

Here $R(1)$ is the size of one step. Associating $R(1)$ with the Compton length $h/m_n c$, we recognize the *Eddington-Weinberg formula* for interpretation of the aggregation process for structures observed in the Universe. Estimating the number of nucleons in a galaxy as 10^{68}, we arrive at the galaxy radius $R \simeq 1-10$ kpc and plausible interrelations for some other cosmic structures (see Table 0.1).

Table 0.1. $N - R$ interrelations for some structures.

Structure	Number of nucleons N	Evaluated size R
Galaxies	10^{68}	1–10 kpc
Clusters of galaxies	10^{72}	1–10 Mpc
Super-clusters of galaxies	10^{73}	10–100 Mpc

Perhaps, this is the most impressive case of scaling relations covering all from vanishingly small particles to unimaginably huge cosmical systems. To express his feelings generated by all-swallowing self-similarity of cosmic

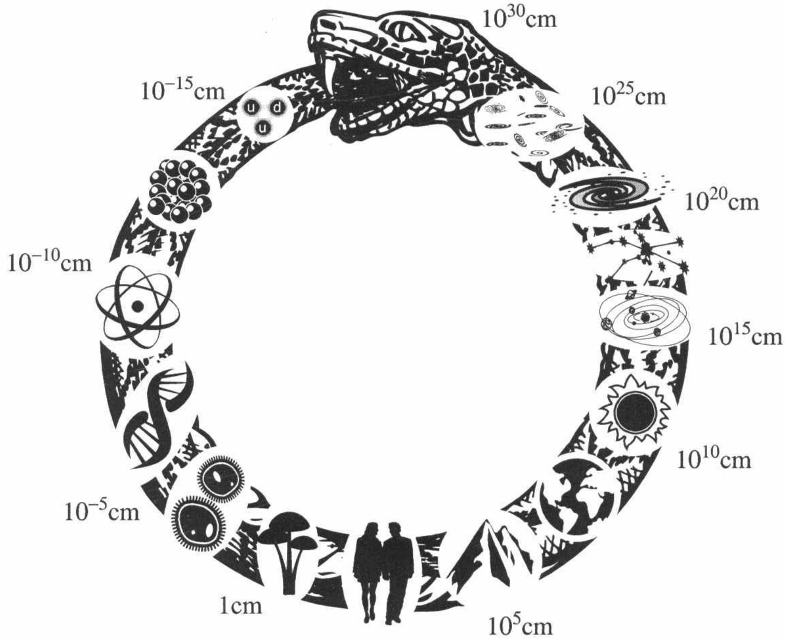

Fig. 0.2 Cosmic Uroboros.

structures, Sheldon Glashow, 1979 Nobel Laureate in Physics, resorted to the image of the *Uroboros*, laid on him some sort of dial with time-scale replaced by length-scale numbered from 10^{-30} cm to 10^{30} cm and named this image the *Cosmic Uroboros*[1] (Fig. 0.2).

Following the self-similarity idea, the outstanding French mathematician Paul Lévy powerfully pushed limits of probability theory by discovering a new class of laws and processes bearing now his name. The ***Lévy motion*** propagator has the form

$$p(x,t;\alpha) = t^{-1/\alpha} g(xt^{-1/\alpha};\alpha), \quad -\infty < x < \infty,$$

with positive constant $\alpha \in (0,2]$ called the *Lévy exponent.* The case $\alpha = 2$ recovers the Brownian motion with Gaussian propagator, but when $\alpha < 2$ we have the whole family of propagators – ***stable Lévy distributions*** – with infinite variances and long tails of inverse power type. The corresponding probability packets expand more rapidly than in the Brownian motion;

[1]The image of a serpent has led many cultures to associate it symbolically with the creation of the world and the unity of all things, especially when the serpent is represented as swallowing its own tail.