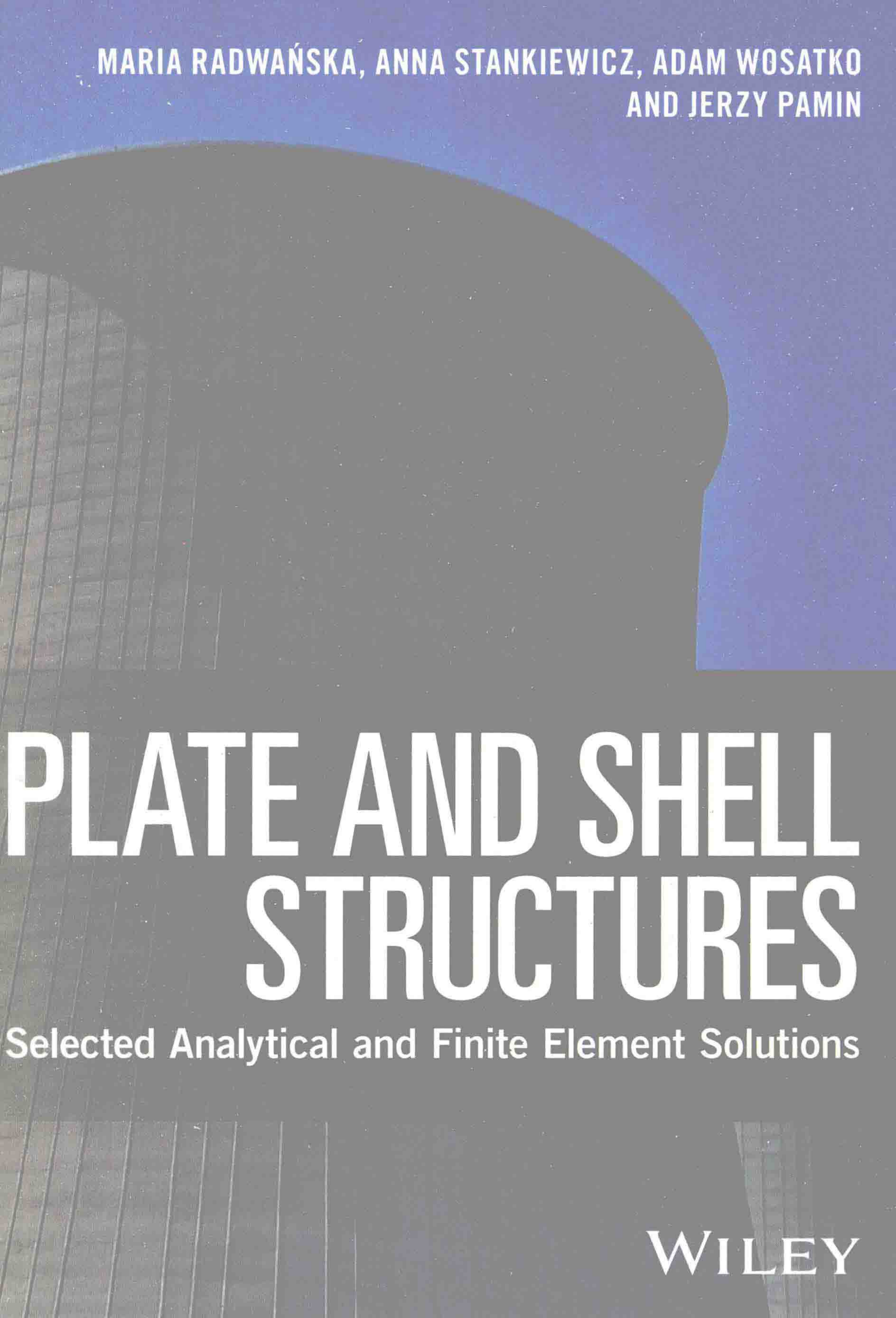

MARIA RADWAŃSKA, ANNA STANKIEWICZ, ADAM WOSATKO AND JERZY PAMIN

PLATE AND SHELL STRUCTURES

Selected Analytical and Finite Element Solutions

WILEY

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Plate and Shell Structures

To our families

Preface

This book deals with the mechanics and numerical simulations of plates and shells, which are flat and curved thin-walled structures, respectively (called shell structures for short in this book). They have very important applications as complete structures or structural elements in many branches of engineering. Examples of shell structures in civil and mechanical engineering include slabs, vaults, roofs, domes, chimneys, cooling towers, pipes, tanks, containers and pressure vessels; in shipbuilding – ship and submarine hulls, in the vehicle and aerospace industries – automobile bodies and tyres and the wings and fuselages of aeroplanes.

The scope of the book is limited to the presentation of the theory of elastic plates and shells undergoing small deformation (thus assuming linear constitutive and kinematic equations).

The book is aimed at the large international community of engineering students, university teachers, professional engineers and researchers interested in the mechanics of shell structures, as well as developers testing new simulation software. The book can be the basis of an intermediate-level course on (computational) mechanics of shell structures at the level of doctoral, graduate and undergraduate studies. The reader should have the basic knowledge of the strength of materials, theory of elasticity, structural mechanics and FEM technology; basic information in these areas is not repeated in the book.

The strength of the book results from the fact that it not only provides the theoretical formulation of fundamental problems of mechanics of plates and shells, but also several examples of analytical and numerical solutions for different types of shell structures. The book also contains some advanced aspects related to the stability analysis and a brief description of classical and modern finite element formulations for plates and shells, including the discussion of mixed/hybrid models and so-called locking phenomena.

The book contains a comprehensive presentation of the theory of elastic plates and shells, formulations and solutions of fundamental mechanical problems (statics, stability, free vibrations) for these structures using exact approaches and computational (approximate) methods, with emphasis on modern capabilities of the finite element (FE) technology. In the book we introduce a large number of examples that illustrate various physical phenomena associated with the behaviour of shell structures under external actions. Comparisons of analytical and numerical solutions are given for several benchmark tests. The book includes plenty of boxes and tables that contain sets of formulae or data and check values describing the examples. They help the reader to find and integrate the information provided and draw conclusions.

The authors are researchers and teachers from the Institute for Computational Civil Engineering of Cracow University of Technology. They have done research on structural mechanics for years, in particular on the theories and advanced computational methods for shell structures, and they also have a long history of teaching the subject to students and practitioners. The selection of the contents of the book is based on this experience. The motivation to write the present book has also come from the fact that there are no books that contain, in one volume, the foundation of the theory and solutions of selected problems using simultaneously analytical and numerical methods.

Following a sequence of subjects: mathematics, theoretical mechanics, strength of materials, structural mechanics, computer science, numerical methods and the finite element method – we have developed a comprehensive course on the mechanics of shell structures. This course contains: (i) discussion of the assumptions and limits of applicability of selected theories on which mathematical models are based, (ii) choice of a method to solve the problem efficiently, (iii) analytical and/or numerical calculations simulating physical phenomena or processes, (iv) confrontation of the results of theoretical and numerical analysis and (v) evaluation of the calculation methodology and results.

Maria Radwańska and Jerzy Pamin were members of Professor Zenon Waszczyszyn's research team, who implemented the finite element code ANKA for buckling and nonlinear analysis of structures at the end of the twentieth century. This resulted in the 1994 Elsevier book: Waszczyszyn, Z. and Cichoń, Cz. and Radwańska, M., *Stability of Structures by Finite Element Method*.

Next, we briefly describe the contents of the book, which is divided into five parts. Part 1 is the introductory part that gives a compact encyclopedic overview of the fundamentals of the theory and modelling of plates and shells in the linear elastic range. A description of static analysis of (plane) plates is contained in Part 2 and of (curved) shells in Part 3. Part 4 includes information on the selected problems of buckling and free vibrations of shell structures. In Part 5, the authors discuss the general aspects of finite element analysis, including the modelling process, evaluation of the quality of finite elements and accuracy of solutions, Part 5 also contains a brief presentation of advanced formulations of finite elements for plates and shells.

While working on the book, we felt special gratitude to two of our teachers: Professors Zenon Waszczyszyn and Michał Życzkowski, who we always thought of as scientific authorities in the field of structural mechanics. In particular, we are deeply indebted to Professor Zenon Waszczyszyn for his invaluable contribution to our knowledge, motivation to do research and to participate in high-level university education. Under his guidance we got to know the theory of plates and shells, computational mechanics applied in civil engineering and modern numerical methods; in particular, the finite element method.

The authors wish to express their appreciation to several colleagues from the Institute for Computational Civil Engineering for discussions and help during the preparation of the book, in particular to A. Matuszak, E. Pabisek, P. Pluciński, R. Putanowicz and T. Żebro. We also record our gratitude to our students who cooperated with us in the computation of numerous examples: M. Abramowicz, M. Bera, I. Bugaj, M. Florek, S. Janowiak, A. Kornaś and K. Kwinta.

Notation

Detailed notation for theoretical analysis

Indices

$\alpha, \beta = 1, 2$	Greek indices (for curvature lines and surface coordinates)
$i, j, k = 1, 2, 3$	Latin indices (for 3D space)
n, m, t	indices for membrane, bending, transverse shear states
$j = 0, 1, 2, \ldots$	number of a components of trigonometric series or number of circumferential wave (half-wave)
(i, j)	indices describing number of waves of deformation in two directions

Coefficients and variables

α_T	coefficient of thermal expansion of a material
A_α: A_1, A_2	Lame coefficients
$\hat{\mathbf{b}} = [\hat{b}_x, \hat{b}_y, \hat{b}_z]^{\mathrm{T}}$	prescribed body forces
$b_{\alpha\beta}$	components of II (second) metric tensor
$\beta = \sqrt{\frac{1}{R\,h}}\,\sqrt[4]{3(1-\nu^2)}$	coefficient in equation of local bending state in cylindrical shell
C^0, C^*	initial and current configuration of a body (shell)
$\mathbf{C}, \mathbf{E} = \mathbf{C}^{-1}$	matrices of local flexibility and stiffness in constitutive equations
$D^n = \dfrac{Eh}{1-\nu^2}$	cross-sectional stiffness in membrane state
$D^m = \dfrac{Eh^3}{12(1-\nu^2)}$	cross-sectional stiffness in bending state
$D^t = \dfrac{k\,Eh}{2(1+\nu)}, k = \dfrac{5}{6}$	cross-sectional stiffness in transverse shear state
$(\mathbf{e}_\alpha, \mathbf{n}), (\mathbf{e}_\alpha^{(z)}, \mathbf{n})$	local base versors on middle surface and on equidistant surface from the middle surface in initial configuration
$(\mathbf{e}_\alpha^*, \mathbf{n}^*)$	local base versors on middle surface in current configuration
$\mathbf{e}^{\mathrm{T}} = [\mathbf{e}^n, \mathbf{e}^m, \mathbf{e}^t]$	generalized strain vector (membrane, bending and transverse shear components)

$\mathbf{e}^n = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^{\mathrm{T}}$	membrane strain vector
$\varepsilon_{11}, \varepsilon_{22}, \gamma_{12} = \varepsilon_{12} + \varepsilon_{21}$	membrane strains: normal and shear in middle surface
$\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta}$	membrane strains in cylindrical system
$\varepsilon_\varphi, \varepsilon_\theta, \gamma_{\varphi\theta}$	membrane strains in spherical system
$\mathbf{e}^m = [\kappa_{11}, \kappa_{22}, \chi_{12}]^{\mathrm{T}}$	bending strain vector
$\kappa_{11}, \kappa_{22}, \chi_{12} = \kappa_{12} + \kappa_{21}$	bending strains: changes of curvature and warping of middle surface
$\kappa_x, \kappa_\theta, \chi_{x\theta}$	bending strains in cylindrical system
$\kappa_\varphi, \kappa_\theta, \chi_{\varphi\theta}$	bending strains in spherical system
$\mathbf{e}^t = [\gamma_{1z}, \gamma_{2z}]^{\mathrm{T}}$	transverse shear strain vector
$\gamma_{xz}, \gamma_{\theta z}$	transverse shear strains in cylindrical system
$\gamma_{\varphi z}, \gamma_{\theta z}$	transverse shear strains in spherical system
$E, \nu, G = E/(2+2\,\nu)$	material constants: Young's modulus, Poisson's ratio, Kirchhoff's modulus
f	rise of shallow shell
F	Airy's stress function
$\mathbf{i}_k$	base versors related to Cartesian coordinates x^k
$g_{\alpha\beta}$	components of I (first) metric tensor
h	thickness of shell
K	Gaussian curvature of surface
$\lambda = \pi/\beta$	length of half-wave for exponential-trigonometric function in local membrane-bending state of cylindrical shell
$m_{11}, m_{22}, m_{12} = m_{21}$	moments: bending and twisting in middle surface
$m_x, m_\theta, m_{x\theta}$	moments in cylindrical system
$m_\varphi, m_\theta, m_{\varphi\theta}$	moments in spherical system
$n_{11}, n_{22}, n_{12} = n_{21}$	membrane forces: normal and tangential in middle surface
$n_x, n_\theta, n_{x\theta}$	membrane forces in cylindrical system
$n_\varphi, n_\theta, n_{\varphi\theta}$	membrane forces in spherical system
$n_{\mathrm{I}}, n_{\mathrm{II}}, m_{\mathrm{I}}, m_{\mathrm{II}}$	principal membrane forces and bending moments
$\tilde{n}_{\nu s}, \tilde{t}_\nu$	effective boundary forces (tangential membrane and transverse shear)
$n_\nu, \tilde{n}_{\nu s}, \tilde{t}_\nu, m_\nu$	generalized boundary forces
$\hat{n}_\nu, \hat{n}_{\nu s}, \hat{t}_\nu, \hat{m}_\nu$	prescribed generalized boundary loads
ν, s, n	directions of boundary base vectors
$\hat{\mathbf{p}} = [\hat{p}_1, \hat{p}_2, \hat{p}_n]^{\mathrm{T}}$	vector of prescribed surface loads
$\hat{\mathbf{p}}_b, \hat{\mathbf{u}}_b$	vectors of prescribed generalized boundary loads and displacements
$\hat{P}_i$	prescribed concentrated force in corner i
$\Pi, \Pi^{(z)}$	middle and equidistant surfaces in initial configuration
$\Pi^*, \Pi^{*\,(z)}$	middle and equidistant surfaces in current configuration

Π_ν, Π_s	cross-sectional planes: normal and tangent to middle surface
Π_α: Π_1, Π_2	two transverse cross-sectional planes normal to middle surface
Π, U, W	total potential energy, internal energy, external load work
$r = f(x)$, $r = f(\varphi)$	meridian equation for axisymmetric shell
R_α: R_1, R_2	principal curvature radii for middle surface of a shell
s	arch coordinate for a line on surface
$\mathbf{s}^{\mathrm{T}} = [\mathbf{s}^n, \mathbf{s}^m, \mathbf{s}^t]$	vector of generalized resultant forces for membrane, bending and transverse shear states
$\mathbf{s}_b = [n_\nu, \tilde{n}_{\nu s}, \tilde{t}_\nu, m_\nu]^{\mathrm{T}}$	vector of generalized boundary forces
$\hat{\mathbf{s}}_b = [\hat{n}_\nu, \hat{n}_{\nu s}, \hat{t}_\nu, \hat{m}_\nu]^{\mathrm{T}}$	vector of presribed boundary forces
$\mathbf{s}^n = \mathbf{n} = [n_{11}, n_{22}, n_{12}]^{\mathrm{T}}$	vector of membrane forces
$\mathbf{s}^m = \mathbf{m} = [m_{11}, m_{22}, m_{12}]^{\mathrm{T}}$	vector of bending and twisting moments
$\mathbf{s}^t = \mathbf{t} = [t_1, t_2]^{\mathrm{T}}$	vector of transverse shear forces
$\varsigma = \sqrt{\dfrac{R}{h}}\,\sqrt[4]{3(1-\nu^2)}$	coefficient in equation of local bending state in spherical shell
T_i	effective force in a corner used in static boundary conditions
$\vartheta = [\vartheta_1, \vartheta_2, \vartheta_n]^{\mathrm{T}}$	vector of rotations
ϑ_α: ϑ_1, ϑ_2	two rotations of normal to middle surface
ϑ_n	rotation around normal to middle surface
$\vartheta_x = \varphi_y, \vartheta_y = -\varphi_x$	two rotations of normal to middle plane of plate under bending in Cartesian system (two alternative notations)
$\sigma_{nn}, \sigma_{ns}, \sigma_{nz}$	stresses: in-plane normal, in-plane tangential, transverse shear
$u = u_x, v = u_y, w$	translations with respect to local system (x, y, z)
U, V, W	translations with respect to global system (X, Y, Z)
$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2, \vartheta_n]^{\mathrm{T}}$	generalized displacement vector
$\mathbf{u} = [u_1, u_2, w]^{\mathrm{T}}$	translation vector in three-parameter thin shell theory
$\mathbf{u} = [w, \vartheta_1, \vartheta_2]^{\mathrm{T}}$	generalized displacement vector in three-parameter moderately thick plate theory
$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2]^{\mathrm{T}}$	generalized displacement vector in five-parameter moderately thick shell theory
$\mathbf{u}_b = [u_\nu, u_s, w, \vartheta_\nu]$	vector of generalized boundary displacements
$\hat{\mathbf{u}}_b = [\hat{u}_\nu, \hat{u}_s, \hat{w}, \hat{\vartheta}_\nu]$	vector of prescribed generalized boundary displacements
U^n, U^m, U^t	strain energy in membrane, bending and transverse shear states
ξ_α: ξ_1, ξ_2	curvilinear surface coordinates on middle surface $z = 0$
$\xi_\alpha = \text{const.}$	coordinate lines on middle surface
$\xi_1 = x, \xi_2 = y$	Cartesian coordinates
$\xi_1 = \varphi, \xi_2 = \theta$	spherical coordinates

$\xi_1 = r, \xi_2 = \theta$	polar coordinates
$\xi_1 = x, \xi_2 = \theta$	cylindrical coordinates
z	coordinate in direction normal to the middle surface Π (distance of equidistant surface $\Pi^{(z)}$ from middle surface Π, $z = 0$ corresponds to the middle surface Π)
$x^k: x^1, x^2, x^3$	Cartesian coordinates with respect to versors $\mathbf{i}_k$
(X, Y, Z)	Cartesian coordinate system
Ω	problem domain
$\partial\Omega$	boundary of domain
$\partial\Omega_\sigma, \partial\Omega_u$	boundary with prescribed loads and displacements, respectively
T_r	reference temperature
T_0	temperature on middle surface
$\Delta T_0 = T_0 - T_r$	temperature change (independent of z) with respect to reference temperature
$\Delta T_h = \Delta T(h/2) - \Delta T(-h/2)$	temperature difference between limiting shell surfaces $z = \pm h/2$

Detailed notation for numerical analysis

Indices

e	index of finite element (FE)
(ef)	index of interelement boundary
n, $node$	index of FE node

Abbreviations

NNDOF, NEDOF, NSDOF	number of degrees of freedom (dofs) for node, element and structure
NSE	number of FEs in a structure
NEN, NSN	number of FE nodes and of structure nodes
NGP	number of Gauss points

Coefficients and variables

$\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_\sigma, \boldsymbol{\alpha}_\varepsilon$	mathematical dofs for interpolation of displacement, stress, strain fields
$\mathbf{B}^n, \mathbf{B}^m, \mathbf{B}^t$	matrices in kinematic relations for membrane, bending and transverse shear states
$\mathbf{D}^n, \mathbf{D}^m, \mathbf{D}^t$	matrices in constitutive equations for membrane, bending and transverse shear states

$\mathbf{f}^e$, $\mathbf{f}_b^e$	vector of substitute nodal forces which represent loads in FE and on FE boundary
$\mathbf{F}$	global vector of substitute nodal forces after assembly process
$I_{\mathrm{p}}[\mathbf{u}]$, $I_{\mathrm{c}}[\boldsymbol{\sigma}]$	potential and complementary energy functionals
$I_{\mathrm{p,m}}$, $I_{\mathrm{c,m}}$	modified potential and complementary energy functionals
$I_{\mathrm{H\text{-}R}}[\mathbf{u}, \boldsymbol{\sigma}]$	two-field Hellinger–Reissner functional
$I_{\mathrm{H\text{-}W}}[\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}]$	three-field Hu–Washizu functional
$G^{(ef)}[\boldsymbol{\sigma}, \mathbf{u}^{(ef)}]$	component added to functional and associated with the equilibrium of tractions on interelement boundary
$H^{(ef)}[\mathbf{u}, \mathbf{t}^{(ef)}]$	component added to functional and associated with the continuity of displacements on interelement boundary
J, $\det \mathbf{J}$	jacobian, determinant of Jacobi matrix
$\mathbf{k}^{en}$, $\mathbf{k}^{em}$, $\mathbf{k}^{et}$	element stiffness matrix for membrane, bending and transverse shear states
$\mathbf{k}_\sigma^e$	stress stiffness matrix in initial and linearized buckling analysis
$\mathbf{k}_u^e$	displacement stiffness matrix for FE in linearized buckling analysis
$\mathbf{L}$	matrix of differential operators in kinematic strain-displacement equations $\epsilon = \mathbf{L}\ \mathbf{u}$
$\mathbf{N}$	matrix of shape functions used for displacement field approximation
$\mathbf{N}_u$, $\mathbf{N}_\sigma$, $\mathbf{N}_\epsilon$	matrices for approximation of displacement, stress, strain fields in two- or three-field formulation in mixed FEs
$\mathbf{P}^*$, $\mathbf{Q}^*$	vectors of reference loads and displacements for one-parameter loading process
$\mathbf{q}^e = \mathbf{q}_u^e$	element generalized displacement vector for displacement-based FE model
$\mathbf{q}_{node}$	nodal generalized displacement (dof) vector for displacement-based FE model
$\mathbf{q}_u^e$, $\mathbf{q}_\sigma^e$, $\mathbf{q}_\epsilon^e$	vectors of element generalized displacement, stress and strain dofs, respectively, for different FE models
$\mathbf{q}_u^{(ef)}$, $\mathbf{q}_t^{(ef)}$	vectors of generalized displacement or, respectively, traction dofs on interelement boundary
$\mathbf{Q}$	vector of generalized displacements for structure
$\mathbf{u}(\mathbf{x})$, $\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\varepsilon}(\mathbf{x})$	displacement, stress, strain fields approximated within FE domain
$\mathbf{u}^{(ef)}(s)$, $\mathbf{t}^{(ef)}(s)$	displacement and traction function approximated along interelement boundary
$\mathbf{R}_{\mathrm{supp}}$	support reaction vector for structure
ξ, η, ζ	natural normalized dimensionless coordinates
Ω^e, $\partial\Omega^e$	area and boundary of FE
$\partial\Omega^{(ef)}$	interelement boundary

Conversions between imperial and metric system units

Quantity	Imperial units	International System of Units (SI)	
length	1 in.	= 2.54 cm	= 0.0254 m
	1 ft.	= 30.48 cm	= 0.3048 m
area	1 in.^{2}	= 6.45 cm^2	= 0.000645 m^2
	1 ft^2	= 929 cm^2	= 0.0929 m^2
force	1 lb-f = 1 lbf	= 4.45 N	= 0.00445 kN
moment	1 lbf-in.	= 11.31 Ncm	= 0.0001131 kNm
intensity of membrane force	1 lbf/in.	= 1.751 N/cm	= 0.175 kN/m
intensity of moment	1 lbf-in./in.	= 4.45 N/cm/cm	= 0.00445 kNm/m
pressure	1 psi = 1 lbf/in.^{2}	= 0.690 N/cm^2	= 6.90 kN/m^2

Contents