

MARIA RADWAŃSKA, ANNA STANKIEWICZ, ADAM WOSATKO  
AND JERZY PAMIN

# PLATE AND SHELL STRUCTURES

Selected Analytical and Finite Element Solutions

WILEY

# Plate and Shell Structures

Selected Analytical and Finite Element Solutions

*Maria Radwańska*

*Anna Stankiewicz*

*Adam Wosatko*

*Jerzy Pamin*

Cracow University of Technology, Poland

**WILEY**

This edition first published 2017  
© 2017 John Wiley & Sons Ltd

*Registered office*

John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at [www.wiley.com](http://www.wiley.com).

The right of Maria Radwańska, Anna Stankiewicz, Adam Wosatko and Jerzy Pamin to be identified as the authors of this work has been asserted in accordance with the Copyright, Designs and Patents Act 1988.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, except as permitted by the UK Copyright, Designs and Patents Act 1988, without the prior permission of the publisher.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books.

Designations used by companies to distinguish their products are often claimed as trademarks. All brand names and product names used in this book are trade names, service marks, trademarks or registered trademarks of their respective owners. The publisher is not associated with any product or vendor mentioned in this book.

**Limit of Liability/Disclaimer of Warranty:** While the publisher and author have used their best efforts in preparing this book, they make no representations or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. It is sold on the understanding that the publisher is not engaged in rendering professional services and neither the publisher nor the author shall be liable for damages arising herefrom. If professional advice or other expert assistance is required, the services of a competent professional should be sought

*Library of Congress Cataloging-in-Publication data applied for*

ISBN: 9781118934548

A catalogue record for this book is available from the British Library.

Cover image: Jasmin Kämmerer/EyeEm/Gettyimages

Typeset in 10/12pt Warnock Pro by SPi Global, Chennai, India  
Printed and bound in Malaysia by Vivar Printing Sdn Bhd

10 9 8 7 6 5 4 3 2 1

## Plate and Shell Structures

*To our families*

## Preface

This book deals with the mechanics and numerical simulations of plates and shells, which are flat and curved thin-walled structures, respectively (called shell structures for short in this book). They have very important applications as complete structures or structural elements in many branches of engineering. Examples of shell structures in civil and mechanical engineering include slabs, vaults, roofs, domes, chimneys, cooling towers, pipes, tanks, containers and pressure vessels; in shipbuilding – ship and submarine hulls, in the vehicle and aerospace industries – automobile bodies and tyres and the wings and fuselages of aeroplanes.

The scope of the book is limited to the presentation of the theory of elastic plates and shells undergoing small deformation (thus assuming linear constitutive and kinematic equations).

The book is aimed at the large international community of engineering students, university teachers, professional engineers and researchers interested in the mechanics of shell structures, as well as developers testing new simulation software. The book can be the basis of an intermediate-level course on (computational) mechanics of shell structures at the level of doctoral, graduate and undergraduate studies. The reader should have the basic knowledge of the strength of materials, theory of elasticity, structural mechanics and FEM technology; basic information in these areas is not repeated in the book.

The strength of the book results from the fact that it not only provides the theoretical formulation of fundamental problems of mechanics of plates and shells, but also several examples of analytical and numerical solutions for different types of shell structures. The book also contains some advanced aspects related to the stability analysis and a brief description of classical and modern finite element formulations for plates and shells, including the discussion of mixed/hybrid models and so-called locking phenomena.

The book contains a comprehensive presentation of the theory of elastic plates and shells, formulations and solutions of fundamental mechanical problems (statics, stability, free vibrations) for these structures using exact approaches and computational (approximate) methods, with emphasis on modern capabilities of the finite element (FE) technology. In the book we introduce a large number of examples that illustrate various physical phenomena associated with the behaviour of shell structures under external actions. Comparisons of analytical and numerical solutions are given for several benchmark tests. The book includes plenty of boxes and tables that contain sets of formulae or data and check values describing the examples. They help the reader to find and integrate the information provided and draw conclusions.

The authors are researchers and teachers from the Institute for Computational Civil Engineering of Cracow University of Technology. They have done research on structural mechanics for years, in particular on the theories and advanced computational methods for shell structures, and they also have a long history of teaching the subject to students and practitioners. The selection of the contents of the book is based on this experience. The motivation to write the present book has also come from the fact that there are no books that contain, in one volume, the foundation of the theory and solutions of selected problems using simultaneously analytical and numerical methods.

Following a sequence of subjects: mathematics, theoretical mechanics, strength of materials, structural mechanics, computer science, numerical methods and the finite element method – we have developed a comprehensive course on the mechanics of shell structures. This course contains: (i) discussion of the assumptions and limits of applicability of selected theories on which mathematical models are based, (ii) choice of a method to solve the problem efficiently, (iii) analytical and/or numerical calculations simulating physical phenomena or processes, (iv) confrontation of the results of theoretical and numerical analysis and (v) evaluation of the calculation methodology and results.

Maria Radwańska and Jerzy Pamin were members of Professor Zenon Waszczyszyn's research team, who implemented the finite element code ANKA for buckling and non-linear analysis of structures at the end of the twentieth century. This resulted in the 1994 Elsevier book: Waszczyszyn, Z. and Cichoń, Cz. and Radwańska, M., *Stability of Structures by Finite Element Method*.

Next, we briefly describe the contents of the book, which is divided into five parts. Part 1 is the introductory part that gives a compact encyclopedic overview of the fundamentals of the theory and modelling of plates and shells in the linear elastic range. A description of static analysis of (plane) plates is contained in Part 2 and of (curved) shells in Part 3. Part 4 includes information on the selected problems of buckling and free vibrations of shell structures. In Part 5, the authors discuss the general aspects of finite element analysis, including the modelling process, evaluation of the quality of finite elements and accuracy of solutions, Part 5 also contains a brief presentation of advanced formulations of finite elements for plates and shells.

While working on the book, we felt special gratitude to two of our teachers: Professors Zenon Waszczyszyn and Michał Życzkowski, who we always thought of as scientific authorities in the field of structural mechanics. In particular, we are deeply indebted to Professor Zenon Waszczyszyn for his invaluable contribution to our knowledge, motivation to do research and to participate in high-level university education. Under his guidance we got to know the theory of plates and shells, computational mechanics applied in civil engineering and modern numerical methods; in particular, the finite element method.

The authors wish to express their appreciation to several colleagues from the Institute for Computational Civil Engineering for discussions and help during the preparation of the book, in particular to A. Matuszak, E. Pabisek, P. Pluciński, R. Putanowicz and T. Żebro. We also record our gratitude to our students who cooperated with us in the computation of numerous examples: M. Abramowicz, M. Bera, I. Bugaj, M. Florek, S. Janowiak, A. Kornaś and K. Kwinta.

## Notation

### Detailed notation for theoretical analysis

#### Indices

$\alpha, \beta = 1, 2$	Greek indices (for curvature lines and surface coordinates)
$i, j, k = 1, 2, 3$	Latin indices (for 3D space)
$n, m, t$	indices for membrane, bending, transverse shear states
$j = 0, 1, 2, \dots$	number of a components of trigonometric series or number of circumferential wave (half-wave)
$(i, j)$	indices describing number of waves of deformation in two directions

#### Coefficients and variables

$\alpha_T$	coefficient of thermal expansion of a material
$A_a: A_1, A_2$	Lame coefficients
$\hat{\mathbf{b}} = [\hat{b}_x, \hat{b}_y, \hat{b}_z]^T$	prescribed body forces
$b_{\alpha\beta}$	components of II (second) metric tensor
$\beta = \sqrt{\frac{1}{R} \frac{1}{h}} \sqrt[3]{3(1-\nu^2)}$	coefficient in equation of local bending state in cylindrical shell
$C^0, C^s$	initial and current configuration of a body (shell)
$\mathbf{C}, \mathbf{E} = \mathbf{C}^{-1}$	matrices of local flexibility and stiffness in constitutive equations
$D^n = \frac{Eh}{1-\nu^2}$	cross-sectional stiffness in membrane state
$D^m = \frac{Eh^3}{12(1-\nu^2)}$	cross-sectional stiffness in bending state
$D^t = \frac{k Eh}{2(1+\nu)}, k = \frac{5}{6}$	cross-sectional stiffness in transverse shear state
$(\mathbf{e}_a, \mathbf{n}), (\mathbf{e}_a^{(z)}, \mathbf{n})$	local base versors on middle surface and on equidistant surface from the middle surface in initial configuration
$(\mathbf{e}_a^*, \mathbf{n}^*)$	local base versors on middle surface in current configuration
$\mathbf{e}^T = [\mathbf{e}^n, \mathbf{e}^m, \mathbf{e}^t]$	generalized strain vector (membrane, bending and transverse shear components)



$\mathbf{e}^n = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]^T$	membrane strain vector
$\varepsilon_{11}, \varepsilon_{22}, \gamma_{12} = \varepsilon_{12} + \varepsilon_{21}$	membrane strains: normal and shear in middle surface
$\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta}$	membrane strains in cylindrical system
$\varepsilon_\varphi, \varepsilon_\theta, \gamma_{\varphi\theta}$	membrane strains in spherical system
$\mathbf{e}^m = [\kappa_{11}, \kappa_{22}, \chi_{12}]^T$	bending strain vector
$\kappa_{11}, \kappa_{22}, \chi_{12} = \kappa_{12} + \kappa_{21}$	bending strains: changes of curvature and warping of middle surface
$\kappa_x, \kappa_\theta, \chi_{x\theta}$	bending strains in cylindrical system
$\kappa_\varphi, \kappa_\theta, \chi_{\varphi\theta}$	bending strains in spherical system
$\mathbf{e}^t = [\gamma_{1z}, \gamma_{2z}]^T$	transverse shear strain vector
$\gamma_{xz}, \gamma_{\theta z}$	transverse shear strains in cylindrical system
$\gamma_{\varphi z}, \gamma_{\theta z}$	transverse shear strains in spherical system
$E, \nu, G = E/(2 + 2\nu)$	material constants: Young's modulus, Poisson's ratio, Kirchhoff's modulus
$f$	rise of shallow shell
$F$	Airy's stress function
$\mathbf{i}_k$	base versors related to Cartesian coordinates $x^k$
$g_{\alpha\beta}$	components of I (first) metric tensor
$h$	thickness of shell
$K$	Gaussian curvature of surface
$\lambda = \pi/\beta$	length of half-wave for exponential-trigonometric function in local membrane-bending state of cylindrical shell
$m_{11}, m_{22}, m_{12} = m_{21}$	moments: bending and twisting in middle surface
$m_x, m_\theta, m_{x\theta}$	moments in cylindrical system
$m_\varphi, m_\theta, m_{\varphi\theta}$	moments in spherical system
$n_{11}, n_{22}, n_{12} = n_{21}$	membrane forces: normal and tangential in middle surface
$n_x, n_\theta, n_{x\theta}$	membrane forces in cylindrical system
$n_\varphi, n_\theta, n_{\varphi\theta}$	membrane forces in spherical system
$n_1, n_{11}, m_1, m_{11}$	principal membrane forces and bending moments
$\tilde{n}_{vs}, \tilde{t}_v$	effective boundary forces (tangential membrane and transverse shear)
$n_v, \tilde{n}_{vs}, \tilde{t}_v, m_v$	generalized boundary forces
$\hat{n}_v, \hat{n}_{vs}, \hat{t}_v, \hat{m}_v$	prescribed generalized boundary loads
$\mathbf{v}, \mathbf{s}, \mathbf{n}$	directions of boundary base vectors
$\hat{\mathbf{p}} = [\hat{p}_1, \hat{p}_2, \hat{p}_n]^T$	vector of prescribed surface loads
$\hat{\mathbf{p}}_b, \hat{\mathbf{u}}_b$	vectors of prescribed generalized boundary loads and displacements
$\hat{p}_i$	prescribed concentrated force in corner $i$
$\Pi, \Pi^{(z)}$	middle and equidistant surfaces in initial configuration
$\Pi^*, \Pi^{*(z)}$	middle and equidistant surfaces in current configuration

$\Pi_v, \Pi_s$	cross-sectional planes: normal and tangent to middle surface
$\Pi_a: \Pi_1, \Pi_2$	two transverse cross-sectional planes normal to middle surface
$\Pi, U, W$	total potential energy, internal energy, external load work
$r = f(x), r = f(\varphi)$	meridian equation for axisymmetric shell
$R_a: R_1, R_2$	principal curvature radii for middle surface of a shell
$s$	arch coordinate for a line on surface
$\mathbf{s}^T = [\mathbf{s}^n, \mathbf{s}^m, \mathbf{s}^t]$	vector of generalized resultant forces for membrane, bending and transverse shear states
$\mathbf{s}_b = [n_v, \tilde{n}_{vs}, \tilde{t}_v, m_v]^T$	vector of generalized boundary forces
$\hat{\mathbf{s}}_b = [\hat{n}_v, \hat{n}_{vs}, \hat{t}_v, \hat{m}_v]^T$	vector of prescribed boundary forces
$\mathbf{s}^n = \mathbf{n} = [n_{11}, n_{22}, n_{12}]^T$	vector of membrane forces
$\mathbf{s}^m = \mathbf{m} = [m_{11}, m_{22}, m_{12}]^T$	vector of bending and twisting moments
$\mathbf{s}^t = \mathbf{t} = [t_1, t_2]^T$	vector of transverse shear forces
$\varsigma = \sqrt{\frac{R}{h}} \sqrt{3(1-\nu^2)}$	coefficient in equation of local bending state in spherical shell
$T_i$	effective force in a corner used in static boundary conditions
$\vartheta = [\vartheta_1, \vartheta_2, \vartheta_n]^T$	vector of rotations
$\vartheta_a: \vartheta_1, \vartheta_2$	two rotations of normal to middle surface
$\vartheta_n$	rotation around normal to middle surface
$\vartheta_x = \varphi_y, \vartheta_y = -\varphi_x$	two rotations of normal to middle plane of plate under bending in Cartesian system (two alternative notations)
$\sigma_{mn}, \sigma_{ns}, \sigma_{nz}$	stresses: in-plane normal, in-plane tangential, transverse shear
$u = u_x, v = u_y, w$	translations with respect to local system $(x, y, z)$
$U, V, W$	translations with respect to global system $(X, Y, Z)$
$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2, \vartheta_n]^T$	generalized displacement vector
$\mathbf{u} = [u_1, u_2, w]^T$	translation vector in three-parameter thin shell theory
$\mathbf{u} = [w, \vartheta_1, \vartheta_2]^T$	generalized displacement vector in three-parameter moderately thick plate theory
$\mathbf{u} = [u_1, u_2, w, \vartheta_1, \vartheta_2]^T$	generalized displacement vector in five-parameter moderately thick shell theory
$\mathbf{u}_b = [u_v, u_s, w, \vartheta_v]$	vector of generalized boundary displacements
$\hat{\mathbf{u}}_b = [\hat{u}_v, \hat{u}_s, \hat{w}, \hat{\vartheta}_v]$	vector of prescribed generalized boundary displacements
$U^n, U^m, U^t$	strain energy in membrane, bending and transverse shear states
$\xi_a: \xi_1, \xi_2$	curvilinear surface coordinates on middle surface $z = 0$
$\xi_a = \text{const.}$	coordinate lines on middle surface
$\xi_1 = x, \xi_2 = y$	Cartesian coordinates
$\xi_1 = \varphi, \xi_2 = \theta$	spherical coordinates

$\xi_1 = r, \xi_2 = \theta$	polar coordinates
$\xi_1 = x, \xi_2 = \theta$	cylindrical coordinates
$z$	coordinate in direction normal to the middle surface $\Pi$ (distance of equidistant surface $\Pi^{(z)}$ from middle surface $\Pi$ , $z = 0$ corresponds to the middle surface $\Pi$ )
$x^k: x^1, x^2, x^3$	Cartesian coordinates with respect to versors $\mathbf{i}_k$
$(X, Y, Z)$	Cartesian coordinate system
$\Omega$	problem domain
$\partial\Omega$	boundary of domain
$\partial\Omega_\sigma, \partial\Omega_u$	boundary with prescribed loads and displacements, respectively
$T_r$	reference temperature
$T_0$	temperature on middle surface
$\Delta T_0 = T_0 - T_r$	temperature change (independent of $z$ ) with respect to reference temperature
$\Delta T_h = \Delta T(h/2) - \Delta T(-h/2)$	temperature difference between limiting shell surfaces $z = \pm h/2$

Detailed notation for numerical analysis

Indices

$e$	index of finite element (FE)
$(ef)$	index of interelement boundary
$n, node$	index of FE node

Abbreviations

$NNDOF, NEDOF, NSDOF$	number of degrees of freedom (dofs) for node, element and structure
$NSE$	number of FEs in a structure
$NEN, NSN$	number of FE nodes and of structure nodes
$NGP$	number of Gauss points

Coefficients and variables

$\alpha_u, \alpha_\sigma, \alpha_\epsilon$	mathematical dofs for interpolation of displacement, stress, strain fields
$\mathbf{B}^n, \mathbf{B}^m, \mathbf{B}^t$	matrices in kinematic relations for membrane, bending and transverse shear states
$\mathbf{D}^n, \mathbf{D}^m, \mathbf{D}^t$	matrices in constitutive equations for membrane, bending and transverse shear states

$\mathbf{f}^e, \mathbf{f}_b^e$	vector of substitute nodal forces which represent loads in FE and on FE boundary
$\mathbf{F}$	global vector of substitute nodal forces after assembly process
$I_p[\mathbf{u}], I_c[\boldsymbol{\sigma}]$	potential and complementary energy functionals
$I_{p,m}, I_{c,m}$	modified potential and complementary energy functionals
$I_{H-R}[\mathbf{u}, \boldsymbol{\sigma}]$	two-field Hellinger–Reissner functional
$I_{H-W}[\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}]$	three-field Hu–Washizu functional
$G^{(ef)}[\boldsymbol{\sigma}, \mathbf{u}^{(ef)}]$	component added to functional and associated with the equilibrium of tractions on interelement boundary
$H^{(ef)}[\mathbf{u}, \mathbf{t}^{(ef)}]$	component added to functional and associated with the continuity of displacements on interelement boundary
$J, \det \mathbf{J}$	jacobian, determinant of Jacobi matrix
$\mathbf{k}^{em}, \mathbf{k}^{em}, \mathbf{k}^{et}$	element stiffness matrix for membrane, bending and transverse shear states
$\mathbf{k}_\sigma^e$	stress stiffness matrix in initial and linearized buckling analysis
$\mathbf{k}_u^e$	displacement stiffness matrix for FE in linearized buckling analysis
$\mathbf{L}$	matrix of differential operators in kinematic strain-displacement equations $\boldsymbol{\epsilon} = \mathbf{L} \mathbf{u}$
$\mathbf{N}$	matrix of shape functions used for displacement field approximation
$\mathbf{N}_u, \mathbf{N}_\sigma, \mathbf{N}_\epsilon$	matrices for approximation of displacement, stress, strain fields in two- or three-field formulation in mixed FEs
$\mathbf{P}^*, \mathbf{Q}^*$	vectors of reference loads and displacements for one-parameter loading process
$\mathbf{q}^e = \mathbf{q}_u^e$	element generalized displacement vector for displacement-based FE model
$\mathbf{q}_{node}$	nodal generalized displacement (dof) vector for displacement-based FE model
$\mathbf{q}_u^e, \mathbf{q}_\sigma^e, \mathbf{q}_\epsilon^e$	vectors of element generalized displacement, stress and strain dofs, respectively, for different FE models
$\mathbf{q}_u^{(ef)}, \mathbf{q}_t^{(ef)}$	vectors of generalized displacement or, respectively, traction dofs on interelement boundary
$\mathbf{Q}$	vector of generalized displacements for structure
$\mathbf{u}(\mathbf{x}), \boldsymbol{\sigma}(\mathbf{x}), \boldsymbol{\epsilon}(\mathbf{x})$	displacement, stress, strain fields approximated within FE domain
$\mathbf{u}^{(ef)}(s), \mathbf{t}^{(ef)}(s)$	displacement and traction function approximated along interelement boundary
$\mathbf{R}_{supp}$	support reaction vector for structure
$\xi, \eta, \zeta$	natural normalized dimensionless coordinates
$\Omega^e, \partial\Omega^e$	area and boundary of FE
$\partial\Omega^{(ef)}$	interelement boundary

Conversions between imperial and metric system units

Quantity	Imperial units	International System of Units (SI)	
length	1 in.	= 2.54 cm	= 0.0254 m
	1 ft.	= 30.48 cm	= 0.3048 m
area	1 in. <sup>2</sup>	= 6.45 cm <sup>2</sup>	= 0.000645 m <sup>2</sup>
	1 ft <sup>2</sup>	= 929 cm <sup>2</sup>	= 0.0929 m <sup>2</sup>
force	1 lb-f = 1 lbf	= 4.45 N	= 0.00445 kN
moment	1 lbf-in.	= 11.31 Ncm	= 0.0001131 kNm
intensity of membrane force	1 lbf/in.	= 1.751 N/cm	= 0.175 kN/m
intensity of moment	1 lbf-in./in.	= 4.45 N/cm/cm	= 0.00445 kNm/m
pressure	1 psi = 1 lbf/in. <sup>2</sup>	= 0.690 N/cm <sup>2</sup>	= 6.90 kN/m <sup>2</sup>

## Contents

**Preface** *xvii*

**Notation** *xix*

### **Part 1 Fundamentals: Theory and Modelling** *1*

#### **1 General Information** *3*

1.1 Introduction *3*

1.2 Review of Theories Describing Elastic Plates and Shells *6*

1.3 Description of Geometry for 2D Formulation *9*

1.3.1 Coordinate Systems, Middle Surface, Cross Section, Principal Coordinate Lines *9*

1.3.2 Geometry of Middle Surface *10*

1.3.3 Geometry of Surface Equidistant from Middle Surface *12*

1.3.4 Geometry of Selected Surfaces *13*

1.3.4.1 Spherical Surface *13*

1.3.4.2 Cylindrical Surface *14*

1.3.4.3 Hyperbolic Paraboloid *15*

1.4 Definitions and Assumptions for 2D Formulation *16*

1.4.1 Generalized Displacements and Strains Consistent with the Kinematic Hypothesis of Three-Parameter Kirchhoff–Love Shell Theory *16*

1.4.2 Generalized Displacements and Strains Consistent with the Kinematic Hypothesis of Five-Parameter Mindlin–Reissner Shell Theory *18*

1.4.3 Force and Moment Resultants Related to Middle Surface *18*

1.4.4 Generalized Strains in Middle Surface *20*

1.5 Classification of Shell Structures *21*

1.5.1 Curved, Shallow and Flat Shell Structures *22*

1.5.2 Thin, Moderately Thick, Thick Structures *22*

1.5.3 Plates and Shells with Different Stress Distributions Along Thickness *23*

1.5.4 Range of Validity of Geometrically Linear and Nonlinear Theories for Plates and Shells *23*

References *24*

#### **2 Equations for Theory of Elasticity for 3D Problems** *26*

Reference *30*

<b>3</b>	<b>Equations of Thin Shells According to the Three-Parameter Kirchhoff–Love Theory</b>	<b>31</b>
3.1	General Equations for Thin Shells	31
3.2	Specification of Lamé Parameters and Principal Curvature Radii for Typical Surfaces	38
3.2.1	Shells of Revolution in a Spherical Coordinate System	39
3.2.2	Shells of Revolution in a Cylindrical Coordinate System	40
3.2.3	Shallow Shells Represented by Rectangular Projection	41
3.2.4	Flat Membranes and Plates in Cartesian or Polar Coordinate System	41
3.3	Transition from General Shell Equations to Particular Cases of Plates and Shells	42
3.3.1	Equations of Rectangular Flat Membranes	42
3.3.2	Equations of Rectangular Plates in Bending	43
3.3.3	Equations of Cylindrical Shells in an Axisymmetric Membrane-Bending State	44
3.4	Displacement Equations for Multi-Parameter Plate and Shell Theories	45
3.5	Remarks	47
	References	47
<b>4</b>	<b>General Information about Models and Computational Aspects</b>	<b>48</b>
4.1	Analytical Approach to Statics, Buckling and Free Vibrations	49
4.1.1	Statics of a Thin Plate in Bending	49
4.1.2	Buckling of a Plate	50
4.1.3	Transverse Free Vibrations of a Plate	51
4.2	Approximate Approach According to the Finite Difference Method	51
4.2.1	Set of Algebraic Equations for Statics of a Plate in Bending	52
4.2.2	Set of Homogeneous Algebraic Equations for Plate Buckling	53
4.2.3	Set of Homogeneous Algebraic Equations for Transverse Free Vibrations of a Plate	53
4.3	Computational Analysis by Finite Element Method	54
4.4	Computational Models – Summary	55
	Reference	55
<b>5</b>	<b>Description of Finite Elements for Analysis of Plates and Shells</b>	<b>56</b>
5.1	General Information on Finite Elements	56
5.2	Description of Selected FEs	58
5.2.1	Flat Rectangular Four-Node Membrane FE	58
5.2.2	Conforming Rectangular Four-Node Plate Bending FE	60
5.2.3	Nonconforming Flat Three- and Four-Node FEs for Thin Shells	63
5.2.4	Two-Dimensional Curved Shell FE based on the Kirchhoff–Love Thin Shell Theory	64
5.2.5	Curved FE based on the Mindlin–Reissner Moderately Thick Shell Theory	64
5.2.6	Degenerated Shell FE	65
5.2.7	Three-Dimensional Solid FE for Thick Shells	65
5.2.8	Geometrically One-Dimensional FE for Thin Shell Structures	66
5.3	Remarks on Displacement-based FE Formulation	69
	References	70

**Part 2 Plates 73**

<b>6</b>	<b>Flat Rectangular Membranes</b>	<b>75</b>
6.1	Introduction	75
6.2	Governing Equations	76
6.2.1	Local Formulation	76
6.2.2	Equilibrium Equations in Terms of In-Plane Displacements	78
6.2.3	Principal Membrane Forces and their Directions	78
6.2.4	Equations for a Flat Membrane Formulated using Airy's Stress Function	79
6.2.5	Global Formulation	80
6.3	Square Membrane under Unidirectional Tension	81
6.3.1	Analytical Solution	81
6.3.2	Analytical Solution with Airy's Stress Function	83
6.3.3	Numerical Solution	83
6.4	Square Membrane under Uniform Shear	83
6.4.1	Analytical Solution	83
6.4.2	FEM Results	84
6.5	Pure In-Plane Bending of a Square Membrane	85
6.6	Cantilever Beam with a Load on the Free Side	88
6.6.1	Analytical Solution	88
6.6.2	FEM Results	92
6.7	Rectangular Deep Beams	94
6.7.1	Beams and Deep Beams	94
6.7.2	Square Membrane with a Uniform Load on the Top Edge, Supported on Two Parts of the Bottom Edge – FDM and FEM Results	94
6.8	Membrane with Variable Thicknesses or Material Parameters	97
6.8.1	Introduction	97
6.8.2	Membrane with Different Thicknesses in Three Subdomains – FEM Solution	97
6.8.3	Membrane with Different Material Parameters in Three Subdomains – FEM Solution	99
	References	101
<b>7</b>	<b>Circular and Annular Membranes</b>	<b>102</b>
7.1	Equations of Membranes – Local and Global Formulation	102
7.2	Equations for the Axisymmetric Membrane State	104
7.3	Annular Membrane	105
7.3.1	Analytical Solution	107
7.3.2	FEM Solution	108
	References	109
<b>8</b>	<b>Rectangular Plates under Bending</b>	<b>110</b>
8.1	Introduction	110
8.2	Equations for the Classical Kirchhoff–Love Thin Plate Theory	110
8.2.1	Assumptions and Basic Relations	110
8.2.2	Equilibrium Equation for a Plate Expressed by Moments	116
8.2.3	Displacement Differential Equation for a Thin Rectangular Plate According to the Kirchhoff–Love Theory	116



8.2.4	Global Formulation for a Kirchhoff–Love Thin Plate	117
8.3	Derivation of Displacement Equation for a Thin Plate from the Principle of Minimum Potential Energy	117
8.4	Equation for a Plate under Bending Resting on a Winkler Elastic Foundation	118
8.5	Equations of Mindlin–Reissner Moderately Thick Plate Theory	119
8.5.1	Kinematics and Fundamental Relations for Mindlin–Reissner Plates	119
8.5.2	Global Formulation for Moderately Thick Plates	120
8.5.3	Equations for Mindlin–Reissner Moderately Thick Plates Expressed by Generalized Displacements	122
8.6	Analytical Solution of a Sinusoidally Loaded Rectangular Plate	122
8.7	Analysis of Plates under Bending Using Expansions in Double or Single Trigonometric Series	127
8.7.1	Application of Navier’s Method – Double Trigonometric Series	127
8.7.2	Idea of Levy’s Method – Single Trigonometric Series	130
8.8	Simply Supported or Clamped Square Plate with Uniform Load	131
8.8.1	Results Obtained using DTSM and FEM for a Simply Supported Plate	132
8.8.2	Results Obtained using STSM and FEM for a Clamped Plate	135
8.9	Rectangular Plate with a Uniform Load and Various Boundary Conditions – Comparison of STSM and FEM Results	135
8.10	Uniformly Loaded Rectangular Plate with Clamped and Free Boundary Lines – Comparison of STSM and FEM Results	139
8.11	Approximate Solution to a Plate Bending Problem using FDM	143
8.11.1	Idea of FDM	143
8.11.2	Application of FDM to the Solution of a Bending Problem for a Rectangular Plate	144
8.11.3	Simply Supported Square Plate with a Uniform Load	147
8.11.4	Simply Supported Uniformly Loaded Square Plate Resting on a One-Parameter Elastic Foundation	149
8.12	Approximate Solution to a Bending Plate Problem using the Ritz Method	151
8.12.1	Idea of the Ritz Method	151
8.12.2	Simply Supported Rectangular Plate with a Uniform Load	152
8.13	Plate with Variable Thickness	153
8.13.1	Description of Deformation	154
8.13.2	FEM Results	154
8.14	Analysis of Thin and Moderately Thick Plates in Bending	155
8.14.1	Preliminary Remarks	155
8.14.2	Simply Supported Square Plate with Uniform Load – Analytical and FEM Results	156
8.14.3	Simply Supported Plate with a Concentrated Central Load – Analytical and FEM Solutions	157
	References	159
<b>9</b>	<b>Circular and Annular Plates under Bending</b>	<b>160</b>
9.1	General State	160
9.2	Axisymmetric State	162