NORTH-HOLLAND SERIES IN
APPLIED MATHEMATICS AND MECHANICS

spectra of partial differential operators

Martin SCHECHTER

second edition

NORTH-HOLLAND

SPECTRA OF PARTIAL DIFFERENTIAL OPERATORS

BY

MARTIN SCHECHTER

University of California, Irvine

1986

NORTH-HOLLAND
AMSTERDAM · NEW YORK · OXFORD

ISBN: 0 444 87822 X

First edition 1971
Second edition 1986

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributions for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 Vanderbilt Avenue
New York, N.Y. 10017
U.S.A.

Library of Congress Cataloging in Publication Data

Schechter, Martin.
Spectra of partial differential operators.

(North-Holland series in applied mathematics and mechanics ; v. 14)
Bibliography: p.
Includes index.
1. Partial differential operators. 2. Spectral theory (Mathematics) I. Title. II. Series.
QA329.42.S33 1986 515.7'242 85-20730
ISBN 0-444-87822-X (U.S.)

PRINTED IN THE NETHERLANDS

SPECTRA OF PARTIAL DIFFERENTIAL OPERATORS

NORTH-HOLLAND SERIES IN

APPLIED MATHEMATICS AND MECHANICS

VOLUME 14

NORTH-HOLLAND
AMSTERDAM · NEW YORK · OXFORD

BE"H

To the memory of my father, Joshua, HK"M,
my first and foremost teacher.

To the memory of my mother, Rose, Z"L,
who sacrificed so much for my education.

To my wife, Deborah, who has been my
constant source of inspiration.

To my children, who represent my hope
for the future.

To my people, the People of the Book, who
have cherished learning throughout the ages.

To all of them I humbly dedicate this book.

PREFACE

Roughly speaking, this book is addressed to the problem of describing the spectrum of a linear partial differential operator in $L^p(E^n)$, where $1 \leqslant p \leqslant \infty$ and E^n is Euclidean n-dimensional space. Simple as the problem sounds, the complete solution is far from known. We have tried to assemble a significant amount of material on the topic, but no attempt at completeness can be made at this time.

Much of what was known before 1962 is contained in the book by Glazman [1965]. However, there has been very rapid development since then, and the present volume is devoted primarily to some of these advances.

The underlying impetus for the study of the problem is not new; it stems from the study of the N-particle Hamiltonian

$$H = \sum_{k=1}^{N} \left\{ \frac{h^2}{8\pi^2 m_k} \sum_{j=1}^{3} \left[\frac{\partial}{\mathrm{i}\,\partial x_{3k+j-3}} + b_j(x^{(k)}) \right]^2 + q_k(x^{(k)}) \right. + V(x)$$

on the Hilbert space $L^2(E^{3N})$ (see ch. 10). We believe that the recent surge in activity is due mainly to two factors. First, new mathematical methods have been developed which have become very useful in treating the classical problems. Secondly, new applications have arisen with complicated potentials $V(x)$. An additional factor is the theoretical interest coming from the study of general operators.

We have searched for a specific area which would

a) include many recent advances,
b) illustrate some of the useful mathematical tools,
c) give indication as to what one can expect in general,
d) be within the grasp of a non-specialist or a graduate student.

The theme we have chosen is the following: Given a partial differential operator $P(\mathrm{D})$ with constant coefficients on E^n and a partial differential operator $Q(x, \mathrm{D})$ with variable coefficients, we consider $P(\mathrm{D}) + Q(x, \mathrm{D})$ as an operator on $L^p(E^n)$ (the precise definition of this operator requires care). Our aim is to describe the effect on the spectrum of $P(\mathrm{D})$ which is produced by the addition of the "perturbation" $Q(x, \mathrm{D})$. Our main concern will be to find conditions on $Q(x, \mathrm{D})$ which will insure that the spectrum of $P(\mathrm{D})$ is not "appreciably" disturbed (of course this requires a precise definition as well).

Since many of the topics treated here are of interest to chemists, engineers, mathematicians and physicists, we have attempted to make the book readable to all. For this purpose we have devoted the first three chapters to background material. Chapter 1 is the largest of the three and contains those topics from functional analysis which are used in the book. Most proofs are omitted in this chapter since the material is readily available in books on functional analysis (see ch. 11). One versed in functional analysis will find little new here. The only sections deserving slight special attention are §§3, 4, 6.

The second chapter deals with those classes of function spaces which are used in the book. Here we merely state the theorems; complete proofs would take us too far afield. References and some proofs are given in ch. 11. Chapter 3 collects the relatively few facts from the theory of partial differential operators which are needed.

It may be a bit surprising that the bulk of the basic information required is not from the theory of partial differential operators, but rather from functional analysis. One reason for this is the fact that we have chosen to avoid boundary value problems. This decision is based on several factors. The theory of boundary value problems is developed to a fairly complete degree only for second order operators or for elliptic operators. Since we have tried to treat a large class of non-elliptic operators of arbitrary order, we would have been unable to carry out the program with respect to boundary value problems. Even more important is the fact that boundary value problems involve much more in the way of "hard analysis" devoted to the derivation of "a priori estimates". Detailed analysis of such problems might obscure the essential aspects of the problems. Moreover, those regular boundary value problems which have been investigated do not give results materially different from the corresponding problems in all space. On the other hand, problems in which degeneracy develops at the boundary have not been studied to the degree that can be used here.

The first three chapters are not to be read as such, but merely used as a reference when needed in the main body of the book. The reader can either assume the theorems stated in them or obtain proofs from the references given in ch. 11. Thus in reality the book begins with chapter 4. The reader should begin there and refer to chs. 1–3 only when the need arises.

The main material is contained in chs. 4–10. The general theory for constant coefficient operators is given in ch. 4 while relative compactness is studied in ch. 5. Elliptic operators are considered in ch. 6 and the L^2 theory for operators bounded from below is given in ch. 7. Chapter 8 treats self-adjoint operators while ch. 9 gives a comprehensive theory for second order operators. In ch. 10 we apply this theory to quantum mechanical systems of particles. The last chapter, ch. 11, gives references, background material and discussions of related work.

Each chapter is divided into sections. In reference, ch. 6, §4 means section 4 of chapter 6. Theorems and lemmas are numbered consecutively in the section where they are found without distinguishing between them. Lemma 2.4 of ch. 7 is the fourth lemma or theorem given in §2 of ch. 7. The chapter number is omitted when reference is made to theorems, lemmas or sections of the same chapter.

I would like to thank my students S. Kohn and H. Koller for helpful suggestions and D. Wilamowsky for correcting parts of the manuscript. I am also indebted to Drs. T. Kato, P. Rejto and I. Segal for interesting conversations and F. Brownell for very helpful correspondence. Many thanks are due to Magdalene McNamara, Barbara Morris, Barbara Roberts and Vinnie Sommerich for typing the manuscript so beautifully. Also deserving of thanks are Bobbie Friedman and Florence Schreibstein for their cheerful help.

The research for this book was sponsored in part by the Air Force Office of Scientific Research, Air Force Systems Command, United States Air Force, under AFOSR grant 69-1786. In particular I want to thank Lt. Col. W. Trott and Dr. R. Pohrer for their encouragement.

Finally I come to my wife, Deborah, whose contributions to this effort cannot be fully appreciated.

In addition, I wish to thank Isaac Bulka for helping to correct the proofs.

New York M.S.
November, 1970

PREFACE TO THE SECOND EDITION

We have added more material, incorporated some recent advances and improved some of the sections. Chapter 6 was completely rewritten to allow us to prove inequalities of the form

$$\|Vu\|_q \leqslant C\|u\|_{s,p}, \quad u \in H^{s,p}$$

for much larger classes of functions $V(x)$. We also obtain conditions for multiplication by $V(x)$ to be a compact operator from $H^{s,p}$ to L^q. In addition we have been able to strenghthen considerably our results concerning s-extensions. This leads to much better conditions on $V(x)$ which guarantee that $P(D)+V$ has an s-extension with the same essential spectrum as $P(D)$.

Two sections were added to ch. 5 describing the spectra of operators in L^1 and L^∞. Conditions are given to insure that the essential spectrum is independent of p in L^p, $1 \leqslant p \leqslant \infty$.

We strenghthened some of the theorems in ch. 7 and added two sections to ch. 8 showing ways of estimating negative eigenvalues. Various methods of obtaining bounds for integral operators are explored. In ch. 9 was replaced the theorem on essential self-adjointness by recent stronger results of Kato. Other sections were improved.

We have added many references to the bilbiography, but we could not make any attempts towards completeness. We would have liked to include many more topics. To do so would have expanded the volume considerably and taken it beyond its original scope. We hope to include them in a subsequent volume.

Irvine
November, 1985

Martin Schechter

CONTENTS

FUNCTIONAL ANALYSIS

§1. Banach and Hilbert spaces

In this chapter we give a brief review of those concepts and results of functional analysis which will be used in the book. Proofs will be supplied only when they are not readily available in a text on the subject or when they are very simple. Specific references will be given in ch. 11. For the convenience of the reader we start from the basic definitions. However, it will be difficult to follow this outline without some prior knowledge of the subject.

A *complex Banach space B* is a set of elements $a, b, c, \ldots$ (sometimes called vectors) for which there is defined an operation of addition and multiplication by complex numbers $\alpha, \beta, \gamma, \ldots$ (sometimes called scalars) such that the following statements hold for all vectors and scalars.

1. $a+b \in B$.
2. $\alpha a \in B$.
3. $a+b=b+a$.
4. $a+(b+c)=(a+b)+c$.
5. $\alpha(a+b)=\alpha a+\alpha b$.
6. $(\alpha+\beta)a=\alpha a+\beta a$.
7. $\alpha(\beta a)=(\alpha\beta)a$.
8. There is an element $0 \in B$ such that $a+0=a$ for all $a \in B$.
9. For each $a \in B$ there is an element $-a \in B$ such that $a+(-a)=0$.
10. To each $a \in B$ there corresponds a real number $\|a\|$ such that
    (a) $\|\alpha a\|=|\alpha|\,\|a\|$,
    (b) $\|a\|=0$ if and only $a=0$,
    (c) $\|a+b\| \leqslant \|a\|+\|b\|$,
    (d) if $\{a_n\}$ is a sequence of elements of B such that $\|a_n-a_m\| \to 0$ as $m, n \to \infty$, then there is an element $a \in B$ such that $\|a_n-a\| \to 0$ as $n \to \infty$.

We usually use $a-b$ as an abbreviation of $a+(-b)$, and $a_n \to a$ as an abbreviation of $||a_n - a|| \to 0$. In the latter case we say that the sequence $\{a_n\}$ *converges to a*.

The number $||a||$ is called the *norm* of a. It is non-negative since

$$0 = ||0|| = ||a-a|| \leq ||a|| + ||-a|| = 2||a|| .$$

Statement 10(d) is usually referred to as the completeness property of B. A sequence $\{a_n\}$ satisfying $||a_n - a_m|| \to 0$ as $n, m \to \infty$ is called a *Cauchy sequence*.

A subset S of B is called *closed* if $a \in S$ whenever there is a sequence $\{a_n\}$ of elements of S such that $a_n \to a$. A subset V of B is called a *subspace* if $\alpha a + \beta b$ is in V for all $a, b \in V$ and all scalars α, β. An element a is called a *linear combination* of the elements $a_1, \ldots, a_n$ if there are scalars $\alpha_1, \ldots, \alpha_n$ such that

$$a = \alpha_1 a_1 + \ldots + a_n a_n . \tag{1.1}$$

The set of elements $a_1, \ldots, a_n$ is called *linearly dependent* if there are scalars $\alpha_1, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 a_1 + \ldots + \alpha_n a_n = 0 . \tag{1.2}$$

Otherwise it is called *linearly independent*. For n a non-negative integer, a subspace V is said to be of dimension $< n$ if every set of n vectors is linearly dependent. It is said to be of dimension n if it is of dimension $< n+1$ but not of dimension $< n$. If V is not of dimension n for any finite n, we say that it has infinite dimension. The dimension of a subspace V is sometimes denoted by dim V.

A set of elements $a_1, \ldots, a_n$ contained in a subspace V of B is called a *basis* of V if it is linearly independent and every element $a \in V$ can be expressed in the form (1.1). The following is an obvious consequence of the definition.

Lemma 1.1. *If the dimension of V is n, then every linearly independent set of n elements is a basis of V. Thus V has a basis.*

A subset S of B is called *bounded* if there is a (finite) number M such that $||a|| \leq M$ for all $a \in S$. It is called *compact* if every sequence $\{a_k\}$ of elements of S has a subsequence converging to an element of S.

Lemma 1.2. *A subspace V of B has finite dimension if and only if all of its bounded closed subsets are compact.*

Lemma 1.3. *For any n elements $a_1, \ldots, a_n$ of B, the set of their linear combinations forms a subspace of B of dimension $\leqslant n$.*

A subset S of B is called *dense* if for each $a \in B$ there is a sequence $\{a_n\}$ of elements of S converging to a. If a subspace is not dense, there is a closed subspace containing it which is not the whole of B.

If M, N are subspaces of B such that $M \cap N = \{0\}$, we let $M \oplus N$ denote the set of all vectors $a \in B$ which can be written in the form $b+c$ with $b \in M$ and $c \in N$. Clearly $M \oplus N$ is a subspace of B.

Lemma 1.4. *Suppose M, N, S are subspaces of B such that $M \cap N = M \cap S = \{0\}$. If $S \oplus M \subset N \oplus M$, then* $\dim S \leqslant \dim N$.

Proof. We shall show that for each n, $\dim N < n$ implies $\dim S < n$. Suppose that $\dim N < n$, and let $a_1, \ldots, a_n$ be any n elements of S. We can write

$$a_j = a_{j1} + a_{j2}\,, \qquad 1 \leqslant j \leqslant n\,,$$

where $a_{j1} \in N$ and $a_{j2} \in M$. Since $\dim N < n$, there are scalars $\alpha_1, \ldots, \alpha_n$ not all zero such that

$$\sum_1^n \alpha_j a_{j1} = 0\,.$$

Hence

$$\sum_1^n \alpha_j a_j = \sum_1^n \alpha_j a_{j2} \in M\,.$$

Since $S \cap M = \{0\}$, we have

$$\sum_1^n \alpha_j a_j = 0\,,$$

and the proof is complete.

A collection of elements which satisfy statements 1–10(c) is called a (complex) *normed vector space* (or *normed linear space*). The following theorem is important in many applications.

Theorem 1.5. *If X is a normed vector space, then there exists a Banach space B containing X such that the norm of B coincides on X with the norm of X, and X is a dense subspace of B.*

The Banach space B is called the *completion* of X.

If X and Y are normed vector spaces, we can form their *cartesian product* $X \times Y$ as follows. We consider all ordered pairs $\langle x, y\rangle$ of elements $x \in X$, $y \in Y$ and define

$$\alpha_1\langle x_1, y_1\rangle + \alpha_2\langle x_2, y_2\rangle = \langle \alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2\rangle$$
$$\|\langle x, y\rangle\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}.$$

One checks easily that under these definitions $X \times Y$ becomes a normed vector space. Moreover, if X and Y are Banach spaces, the same is true of $X \times Y$.

A *scalar product* on a (complex) normed vector space X is an assignment which assigns a complex number (x, y) to each element $\langle x, y\rangle$ of $X \times X$ in such a way that

(i) $(\alpha x, y) = \alpha(x, y)$,

(ii) $(x, y) = \overline{(y, x)}$ (the complex conjugate),

(iii) $(x+y, z) = (x, y) + (y, z)$,

(iv) $(x, x) = \|x\|^2$.

A Banach space which has a scalar product is called a *Hilbert* space.

In a Hilbert space X we shall say that elements x, y are *orthogonal* and write $x \perp y$ if $(x, y) = 0$. If S is a subset of X we let $S^{\perp}$ denote the set of all elements of X which are orthogonal to all elements of S. If V is a closed subspace of X, we have $X = V \oplus V^{\perp}$.

A sequence $\{x_k\}$ of elements of a Hilbert space X is said to *converge weakly* to an element $x \in X$ if $(x_n - x, y) \to 0$ as $n \to \infty$ for each $y \in X$. We write $x_n \rightharpoonup x$ in this case. We shall need

Theorem 1.6. *If $\{x_k\}$ is a sequence of elements of a Hilbert space X such that*

$$\|x_k\| \leqslant C, \qquad k = 1, 2, \ldots,$$

then there is a subsequence $\{y_n\}$ of $\{x_k\}$ and an element $z \in X$ such that $y_n \rightharpoonup z$.

A sequence $\{x_k\}$ of elements of a Hilbert space is called *orthonormal* if

$$(x_j, x_k) = \delta_{jk}, \qquad j, k = 1, 2, \ldots,$$

where δ_{jk} is the Kronecker delta ($\delta_{jk} = 0$ for $j \neq k$, $\delta_{kk} = 1$).

Lemma 1.7. *Every infinite dimensional subspace of a Hilbert space contains an orthonormal sequence.*