普通高等教育“十三五”规划教材

Advanced Mathematics

(2nd Edition) (II)

高等数学（第2版）（下）

北京邮电大学高等数学双语教学组 编

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内 容 简 介

本书是根据国家教育部非数学专业数学基础课教学指导分委员会指定的工科类本科数学基础课程教学基本要求编写的全英文教材，全书分为上、下两册，此为下册，主要包括无穷级数、向量与空间解析几何、多元函数微分学、重积分、曲线积分与曲面积分。本书对基本概念的叙述清晰准确，对基本理论的论述简明易懂，例题习题的选配典型多样，强调基本运算能力的培养及理论的实际应用。本书可作为高等理工院校非数学类专业本科生的教材，也可供其他专业选用和社会读者阅读.

The aim of this book is to meet the requirement of bilingual teaching of advanced mathematics. This book consists of two volumes. The second volume contains infinite series, vector and the spatial analytic geometry, the differential calculus of multivariate function, multiple integral, and line integral and surface integral. The selection of the contents is in accordance with the fundamental requirements of teaching issued by Ministry of Education of China. This book may be used as a textbook for undergraduate students in the science and engineering schools whose majors are not mathematics, and may also be suitable to the readers at the same level.

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Preface

Advanced mathematics that we refer to contains mainly calculus. Calculus is the mathematics of motion and change. It was first invented to meet the mathematical needs of the scientists of the sixteenth and seventeenth centuries, and the needs that were mainly mechanical in nature. Differential calculus deals with the problem of calculating rates of change. It enables people to define slopes of curves, to calculate velocities and accelerations of moving bodies etc.. Integral calculus deals with the problem of determining a function from information about its rate of change. It enables people to calculate the future location of a body from its present position and a knowledge of the forces acting on it, to find the areas of irregular regions in the plane, to measure the lengths of curves, and so on. Now advanced mathematics becomes one of the most important courses of the college students in natural science and engineering.

The second edition of the book is revised based on implementation experience of its first edition. The contents of the book are written by the authors as follows: Professor Ping Zhu, Professor Jianhua Yuan, Associate Professor Xiaohua Li and Associate Professor Huixia Mo. All the Chapters of the book is organized and proofread by Professor Wenbao Ai. The new edition is contributed as logically and intuitively as possible. Its Chinese and English versions and a corresponding exercise book form a family-united system, which is very useful to the bilingual-teaching. For any errors remaining in the book, the authors would be grateful if they were sent to: jianhuayuan@bupt.edu.cn.

Authors

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Chapter 7

Infinite Series

The **infinite series** [无穷级数], which will be introduced in this chapter, is an important part of advance mathematics. The infinite series is closely related to the infinite sequence, and it is a new form of expression of the limit of the infinite sequence. Then, it can be studied by the theory of the limit of the series. With the establishment of the theorems for the convergence and divergence of the series, the theory of infinite series has also promoted the development of the limit theory. Infinite series provide a very useful tool for expressing functions, studying properties of functions, and doing some approximate computations.

We will first introduce the concepts and properties of an **infinite series with constant terms** [常数项级数] and then some convergence tests for series with constant terms. Later we use this as a basic for the study of **infinite series with function terms** [函数项级数]. Finally, two important types of series of functions, namely **power series** [幂级数] and **Fourier series** [傅里叶级数], will be investigated.

7.1 Concepts and Properties of Series with Constant Terms

7.1.1 Examples of the Sum of an Infinite Sequence

Example 7.1.1 (Length of perpendiculars) A right triangle ABC is given with $\angle A=\theta$, $\angle C=\frac{\pi}{2}$ and $|AC|=b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, EF is drawn perpendicular to AB, and this process is continued indefinitely as shown in Figure 7.1.1. Find the total length of all the perpendiculars

$$|CD|+|DE|+|EF|+|FG|+\cdots$$

in terms of b and θ.

Solution According to the assumption, the process of generating of perpendiculars is continued indefinitely. It is easy to see that when $\angle A=\theta$ and $|AC|=b$, the length of the first perpendicular CD is

$$L_0=|CD|=b\sin\theta;$$

The length of the second perpendicular DE is

$$L_1=|DE|=b\sin^2\theta;$$

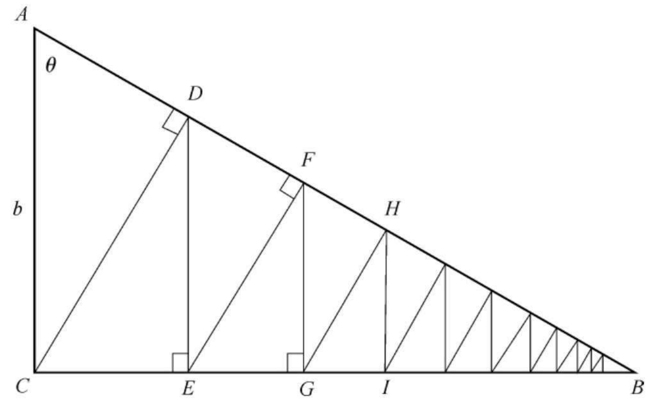

Figure 7. 1. 1

The length of the third perpendicular EF is

$$L_2=|EF|=b\sin^3\theta;$$

$$\vdots$$

The length of the nth perpendicular is

$$L_n=b\sin^{(n+1)}\theta.$$

Hence, the total length L of all the perpendiculars is

$$L=b\sin\theta+b\sin^2\theta+b\sin^3\theta+\cdots+b\sin^{(n+1)}\theta+\cdots. \tag{7.1.1}$$

■

Example 7. 1. 2 (Problem of a ball with bounce) Drop a ball from H meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less 1.

(1) Assuming that the ball continuous to bounce indefinitely, find the total distance of the ball's travel;

(2) Calculate the total time of the ball's travel. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)

Solution (1) The distance of the ball to fall down from height H to the ground is

$$S_0=H;$$

The distance of the ball to first bounce up and then fall down to the ground again is

$$S_1=2Hr;$$

Repeating the above process yields

$$S_2=2Hr^2;$$

$$\vdots$$

The distance of the ball to bounce up and fall down in the nth time is

$$S_n=2Hr^n.$$

Therefore, the total distance of the ball travels up and down is

$$S=H+2Hr+2Hr^2+\cdots+2Hr^n+\cdots. \tag{7.1.2}$$

(2) By $h=\frac{1}{2}gt^2$, the time of the first fall of the ball from height H to the ground is

$$T_0=\sqrt{\frac{2H}{g}}.$$

Similarly, the time for the ball to first bounce up from the ground to the height rH and then fall down to the ground again is

$$T_1=2\sqrt{\frac{2rH}{g}}.$$

In general, the time for the ball to bounce up and fall down in the nth time is

$$T_n=2\sqrt{\frac{2r^nH}{g}}.$$

Thus, the total time of the ball's travel is

$$T=\sqrt{\frac{2H}{g}}+2\sqrt{\frac{2rH}{g}}+2\sqrt{\frac{2r^2H}{g}}+\cdots+2\sqrt{\frac{2r^nH}{g}}+\cdots. \qquad (7.1.3)$$ ■

From the two examples, we encounter the problem of the "sum" of an infinite number of values. Different from the sums with finite terms, the sums with infinite terms sometimes make no sense, that is, it may not correspond to a number. Therefore, the first question is what the meaning of the sum of an infinite sequence is.

7.1.2 Concepts of Series with Constant Terms

Definition 7.1.3(Infinite series［无穷级数］) Suppose that there is an infinite sequence of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$, then we write its sum as

$$a_1+a_2+a_3+\cdots+a_n+\cdots. \qquad (7.1.4)$$

Thus the expression of the sum is called a **series of constant terms**［常数项级数］or **infinite series**［无穷级数］(simply a **series**［级数］), where a_n is called the **general term**［通项］of the series or the nth term of the series.

The series of constant terms (7.1.4) can be denoted by the sigma notation

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n.$$

As we know, addition of real numbers is a binary operation. This means that we really add two numbers at a time. The only reason that $1+2+3$ makes sense as "addition" is that we can group the numbers and then add them two at a time. In short, a finite sum of real numbers always produces a real number, but an infinite sum of real numbers is something else entirely. This is why we need a careful definition of infinite series.

It would be impossible to find a finite sum for series

$$1+2+3+4+\cdots+n+\cdots,$$

because if we start adding the terms, we get the cumulative sums $1, 3, 6, 10, 15, \cdots$, and after the nth term, we get $n(n+1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots,$$

we get the cumulative sums

$$\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\cdots,1-\frac{1}{2^n},\cdots.$$

As we add more and more terms, these sums become closer and closer to 1. In fact, by adding sufficiently many terms of the series, we can make the sums as close as we like to 1 (See this by adding the areas in the "infinitely halved" unit square, Figure (7.1.2)). So, it seems the sum of this infinite series is 1, namely

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots=1.$$

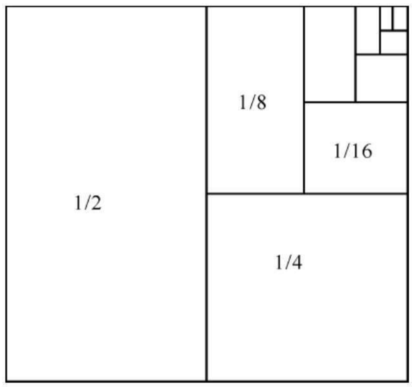

Figure 7.1.2

In the following, we will use the similar idea to determine whether a series is convergent or divergent.

Generally, the sum of the first n terms of the series $S_n=\sum_{k=1}^{n}a_n(n=1,2,\cdots)$ is called the ***n*th partial sum** [前 n 项部分和] of the series or simply **partial sum** [部分和]. The partial sums of the series as follows

$$\begin{aligned}
S_1 &= a_1\\
S_2 &= a_1+a_2\\
S_3 &= a_1+a_2+a_3\\
&\vdots\\
S_n &= a_1+a_2+a_3+\cdots+a_n=\sum_{k=1}^{n}a_k\\
&\vdots
\end{aligned}$$

form a sequence, which is called a **sequence of partial sums** of the series. We can use whether or not the sequence of partial sums has a limit as $n\to\infty$ to determine whether or not a general series has a sum.

Definition 7.1.2 If the sequence of partial sums $\{S_n\}$ converges, then we say that the series $\sum_{n=1}^{\infty}a_n$ **converges** [收敛]. In this case, the limit of the sequence $\{S_n\}$, $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sum_{k=1}^{n}a_n=S$

is called the **sum** of the series, denoted by $\sum_{n=1}^{\infty} a_n = S$. Otherwise, we say that the series **diverges**［发散］. The convergence or divergence of a series can be referred to as convergence property.

The difference between the sum and the partial sum of the series, $R_n = S - S_n = \sum_{k=n+1}^{\infty} a_k$ is called the ***n*th remainder**［余项］of the series.

Example 7.1.3 (Geometric series［几何级数］) Discuss the convergence of the **series of equal ratios**［等比级数］(or **geometric series**)

$$a + aq + aq^2 + \cdots + aq^{n-1} = \sum_{n=0}^{\infty} aq^n \quad (a \neq 0), \tag{7.1.5}$$

where q is called the **common ratio of the series.**

Solution (1) When $q=1$, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty$, so the series is divergent.

(2) When $q=-1$, the given series become $a-a+a-a+\cdots+(-1)^{n-1}a+\cdots$.

Since

$$S_n = \begin{cases} a, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases}$$

so the series is divergent.

(3) When $|q| \neq 1$, the partial sum of the given series is

$$a + aq + aq^2 + \cdots + aq^{n-1} = \frac{a(1-q^n)}{1-q}.$$

If $|q|<1$, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \frac{a}{1-q}$, so the series is convergent and its sum is $\frac{a}{1-q}$.

If $|q|>1$, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \infty$, so the series is divergent.

We summarize the results of the above example as follows.

The series (7.1.5) is convergent when $|q|<1$, and its sum is $\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}$; if $|q| \geqslant 1$, the geometric series is divergent. ■

Now, let's solve the Examples 7.1.1 and 7.1.2 by the results of Example 7.1.3.

For the perpendiculars of triangle of Example 7.1.1, since the expression (7.1.1) is just the geometric series with ratio $r=\sin\theta$, the total length of all the perpendiculars is

$$L = \frac{b\sin\theta}{1-\sin\theta}.$$

For the ball's travel of Example 7.1.2, the expression (7.1.2) is just the geometric series with ratio r (from the second term) and the expression (7.1.3) is just the geometric series with ratio $\sqrt{r}$ (from the second term), where $0<r<1$. So, the travel distance is

$$S = H + \frac{2Hr}{1-r},$$

and the travel time is

$$T = \sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2rH}{g}}\frac{1}{1-\sqrt{r}} = \sqrt{\frac{2H}{g}}\left(\frac{1+\sqrt{r}}{1-\sqrt{r}}\right).$$ ■

Example 7.1.4 Find the sum of the following series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{n(n+1)}+\cdots.$$

Solution Since $\quad a_k=\dfrac{1}{k(k+1)}=\dfrac{1}{k}-\dfrac{1}{k+1}\quad (k=1,2,\cdots),$

then the ***n*th partial sum** is

$$S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\frac{1}{n+1}.$$

Since $\lim\limits_{n\to\infty}S_n=1$, then the series is convergent and its sum is 1. ■

Example 7.1.5 Find the sum of the series $\sum\limits_{n=1}^{\infty}\arctan\dfrac{1}{2n^2}$.

Solution Using the formula $\arctan x-\arctan y=\arctan\dfrac{x-y}{1+xy}\quad (x>0,y>0)$,

we obtain that

$$a_k=\arctan\frac{1}{2k^2}=\arctan\frac{1}{2k-1}-\arctan\frac{1}{2k+1}\quad (k=1,2,\cdots).$$

So,

$$\begin{aligned}S_n&=\sum_{k=1}^{n}\arctan\frac{1}{2k^2}\\&=\left(\arctan 1-\arctan\frac{1}{3}\right)+\left(\arctan\frac{1}{3}-\arctan\frac{1}{5}\right)+\cdots+\left(\arctan\frac{1}{2n-1}-\arctan\frac{1}{2n+1}\right)\\&=\arctan 1-\arctan\frac{1}{2n+1}.\end{aligned}$$

Since $\lim\limits_{n\to\infty}S_n=\dfrac{\pi}{4}$, then $\sum\limits_{n=1}^{\infty}\arctan\dfrac{1}{2n^2}=\dfrac{\pi}{4}$. ■

Example 7.1.6 Express the repeating decimal 5.232 323 ⋯as the ratio of two integers.

Solution The repeating decimal 5.232 323 ⋯can be written as the following,

$$5.\dot{2}\dot{3}=5+\frac{23}{100}+\frac{23}{100^2}+\frac{23}{100^3}+\frac{23}{100^4}+\cdots.$$

Since $\dfrac{23}{100}+\dfrac{23}{100^2}+\dfrac{23}{100^3}+\dfrac{23}{100^4}+\cdots$is a geometric series with ratio $q=\dfrac{1}{100}$. Therefore,

$$\begin{aligned}5.232\,323\cdots&=5+\frac{23}{100}+\frac{23}{100^2}+\frac{23}{100^3}+\frac{23}{100^4}+\cdots\\&=5+\frac{23}{100}\left(1+\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\cdots\right)\\&=5+\frac{23}{100}\times\frac{100}{99}\\&=5+\frac{23}{99}=\frac{518}{99}.\end{aligned}$$ ■

7.1.3 Properties of Series with Constant Terms

From the above section, we see that the convergence of the series corresponds to the limit of the sequence of its partial sums. Thus, the following properties of series follow exactly those of sequences.

Theorem 7.1.1 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge and their sums are S and $\overline{S}$ respectively, then

(1) **(Properties of linear)** For any $\alpha, \beta \in \mathbf{R}$, the series $\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n)$ also converges and

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n = \alpha S + \beta \overline{S};$$

(2) If $a_n \leqslant b_n$ for any $n \in \mathbf{N}_+$, then $\sum_{n=1}^{\infty} a_n \leqslant \sum_{n=1}^{\infty} b_n$.

Corollary 7.1.1 If the series $\sum_{n=1}^{\infty} a_n$ is convergent, and the series $\sum_{n=1}^{\infty} b_n$ is divergent, then the series $\sum_{n=1}^{\infty} (a_n \pm b_n)$ is divergent.

Example 7.1.7 Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \cos n\pi\right)$.

Solution Since the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, and the series $\sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$ is divergent, then the series $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \cos n\pi\right)$ is divergent. ■

Theorem 7.1.2 If a series converges, then its sum is not changed when we add arbitrarily some brackets among the terms of the series (provided that the order of the terms is maintained).

Proof Let the sequence of the partial sums of a convergent series $\sum_{n=1}^{\infty} a_n$ be $\{S_n\}$. Adding arbitrarily some brackets among the terms of the series, we get a new series

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}) + \cdots + (a_{n_{k-1}+1} + a_{n_{k-1}+2} + \cdots + a_{n_k}) + \cdots. \tag{7.1.6}$$

Let the sequence of the partial sums of series (7.1.6) be $\{\overline{S}_k\}$. We have

$$\overline{S}_1 = S_{n1}, \overline{S}_2 = S_{n_2}, \cdots, \overline{S}_k = S_{n_k}, \cdots,$$

that is, $\{\overline{S}_k\}$ is a subsequence of $\{S_n\}$. Since $\{S_n\}$ is convergent and $\lim_{n\to\infty} S_n = S$, so the subsequence $\{\overline{S}_k\}$ is convergent, too. And, $\lim_{n\to\infty} \overline{S}_k = S$. ■

Notice that the converse of the Theorem 7.1.2 may not be true. For example, the series

$$(1-1)+(1-1)+(1-1)+\cdots+(1-1)+\cdots=0+0+0+\cdots+0+\cdots=0$$

converges, but the original series

$$1-1+1-1+\cdots+(-1)^{n-1}+\cdots$$

is divergent.

Theorem 7.1.3 (Adding or deleting terms) Deleting, adding or changing any finite numbers of terms of the infinite series does not change the convergence or divergence of the series.

Theorem 7.1.4 (Necessary condition for convergence) If a series $\sum_{n=1}^{\infty} a_n$ converges, then

(1) $\lim\limits_{n\to\infty} a_n = 0$;

(2) $\lim\limits_{n\to\infty} R_n = 0$, where R_n is the **nth remainder** of the series $\sum\limits_{n=1}^{\infty} a_n$.

Proof Let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ be the **nth partial sum** of the series, and $\lim\limits_{n\to\infty} S_n = S$.

(1) Since $a_n = S_n - S_{n-1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0;$$

(2) Since $R_n = \sum\limits_{k=n+1}^{\infty} a_k = S - S_n$, then

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(S - S_{n-1}) = S - S = 0.$$ ■

In Theorem 7.1.4, (1) and (2) are both the necessary conditions for the convergent series. However, the condition (1) is easy to check for us, so (1) can be used to verify the divergence of a given series. Therefore, when we determine the convergence or divergence of the series $\sum\limits_{n=1}^{\infty} a_n$, we should check that $\lim\limits_{n\to\infty} a_n = 0$ is true or not. If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent. But, we should remember that $\lim\limits_{n\to\infty} a_n = 0$ doesn't mean that the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

Now, let us consider the series

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2} + \underbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}_{3} + \cdots + \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n} + \frac{1}{n+1} + \cdots.$$

The nth term of the series $a_n = \frac{1}{n} \to 0\ (n\to\infty)$, but the series is divergent. In fact, if the series is convergent, then

$$1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) + \frac{1}{n+1} + \cdots$$

is convergent, too. But the sum of the terms in every bracket is 1 for the new series, so the given series is divergent.

Example 7.1.8 Determine the convergence or divergence of the following series.

(1) $\sum\limits_{n=1}^{\infty} (-1)^{n+1}$; (2) $\sum\limits_{n=1}^{\infty} \frac{n^2}{4n^2 - 3}$.

Solution (1) Let $a_n = (-1)^{n+1}$. It is easy to see that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}(-1)^{n+1}$ doesn't exist, so the series diverges by Theorem 7.1.4.

(2) Let $a_n = \frac{n^2}{4n^2 - 3}$. It is easy to see that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^2}{4n^2 - 3} = \frac{1}{4} \neq 0$, so the series diverges by Theorem 7.1.4. ■

Using the Cauchy convergence criteria to determine the convergence or divergence of the nth partial sum $\{S_n\}$, then we can obtain the Cauchy convergence criteria of the series.

* **Theorem 7.1.5 (Cauchy convergence criteria)** The series $\sum\limits_{n=1}^{\infty} a_n$ is convergent if and only