

Modeling and Simulation in Science,  
Engineering and Technology

Tomás Chacón Rebollo  
Roger Lewandowski

# Mathematical and Numerical Foundations of Turbulence Models and Applications

 Birkhäuser

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Tomás Chacón Rebollo  
Department of Differential Equations  
and Numerical Analysis and Institute of  
Mathematics (IMUS)  
University of Seville  
Seville, Spain

Roger Lewandowski  
Institute of Mathematical Research  
of Rennes, IRMAR - UMR 6625, CNRS  
University of Rennes 1  
Rennes, France

ISSN 2164-3679

ISSN 2164-3725 (electronic)

ISBN 978-1-4939-0454-9

ISBN 978-1-4939-0455-6 (eBook)

DOI 10.1007/978-1-4939-0455-6

Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2014937795

Mathematics Subject Classification (2010): 76F02, 76F05, 76M55, 76F55, 76F60, 76F65, 76F40, 35Q30, 76D05, 76D99, 76M30, 35J50, 35K55, 46E35, 65M12, 65M22, 65N30, 76M25, 76-04.

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*To my wife Mame and my children Carmen  
and Tomás*

*Tomás Chacón Rebollo*

*To my wife Sophie and my children Noé,  
Yann, Sarah, and Marie*

*Roger Lewandowski*

# Preface

This book is about turbulence in incompressible fluids.

We have asked people in the street what the word “turbulence” means for them. One woman replied: “Turbulence makes me think of the sea, because it makes one feel what is invisible, what cannot be predicted.” More generally, people answered giving only one word, such as disorder, aircraft, clouds, weather forecast, power, and chemistry. Therefore, turbulence is something that anyone has experienced in one way or another. Mathematicians will answer that turbulence is about fluids, mixing, chaos, and connected scales. It may be a source of inspiration for painters or poets. One may attempt to control it for technological progress. It is however a source of concern because of its impact on environment and human life, the most critical environmental challenge being climate change.

Although understanding turbulence is of primary importance, there is no mathematical definition of it, and many physical mechanisms governing turbulent motions remain unknown. One could say that there is a chance for mankind to understand quantum physics someday, but not turbulence. Nevertheless, it is possible to simulate by means of computers some features of turbulent motions: weather forecasts are rather accurate over 5 days, the mean Gulf Stream path can be calculated, numerical flow simulations around an aircraft wing are in good agreement with experimental data, etc. All these numerical simulations are performed by means of “turbulence models.”

Turbulence models aim to simulate statistical means of turbulent flows or some of their scales. It is however estimated that an accurate computation of all scales of such flows will be possible only by the end of the twenty-first century, if the improvement of the computational resources continues at the same rate.

We do not pretend to give a definition of what turbulence is. Our goal is to provide a comprehensive and innovative presentation of turbulence models, at the crossroads of modeling and mathematical and numerical analysis, including all these aspects in one single book, in complementarity with the other reference manuals in the field.

This book is the synthesis of almost 20 years of thoughts and works about turbulence models, through the meeting of a mathematician with a numerical analyst, leading to a long-term collaboration and friendship. This resulted in

several joint research works, which gave us the opportunity to check that the complementarity of these specialities can be quite fruitful. Finally, it led us to the project of jointly writing a book from a comprehensive point of view on one of the most challenging scientific problems, as is the understanding of turbulence: we deliver here what we are able to understand from turbulence.

In mathematics, authors are always listed in alphabetical order, which is the case of this book.

Seville, Spain  
Rennes, France  
January 2014

Tomás Chacón Rebollo  
Roger Lewandowski

# Acknowledgements

We were first involved in turbulence by Claude Bardos and Olivier Pironneau several decades ago. Our constant discussions with them throughout these years have been crucial in our understanding of the turbulence models, so that our warmest thanks go to them.

We are especially grateful to Luigi Berselli, Traian Iliescu, William Layton, Olivier Pironneau, and Leo Rebholz, who reviewed parts of this book and suggested numerous corrections.

A particular thank goes to Samuele Rubino, who obtained a large part of the computational results and provided the computer code.

We thank Nicola Bellomo who initiated the project as editor and Ben Cronin, Tom Grasso, Allen Mann, and Mitch Moulton from Birkhäuser who provide us a constant and effective support throughout the editorial process.

We also thank Elsevier and Springer publishing for letting us include some figures in the book.

Seville, Spain  
Rennes, France  
January 2014

Tomás Chacón Rebollo  
Roger Lewandowski

I am deeply grateful to Christine Bernardi, Juan Casado, Vivette Girault, and François Murat, for fruitful discussions on several hard technical points, and also to my sister Tere; my friends from Sevilla Isabel, Macarena, Marga, Pedro, and Toñi; and those from Prado del Rey, Isa, Isa Mari, Jimmy, Juana, and Manolo, for their permanent support during the writing of the book. A very special thought goes to Paco who prematurely passed away and who would have enjoyed so much to see this book finished.

I thank the Departamento de Ecuaciones Diferenciales y Análisis Numérico of the University of Seville, the research group M2S2M, and the Laboratoire Jacques-Louis Lions for their kindness and practical support. Special thanks go to Ara for her help in many administrative tasks. I also express my gratitude to the Spanish Government and the European Union, who partially funded the research that has



led to this book, through the successive grants MTM2006-01275, MTM2009-07719 and MTM2012-36124-C02-01.

Finally I want to express my deepest gratitude to my wife Mame for her permanent support and never-ending patience during the preparation of the manuscript.

Seville, Spain  
January 2014

Tomás Chacón Rebollo

My special thoughts go to Jacques-Louis Lions, who had a significant influence on my scientific evolution between 1994 and 2001 and whose teaching and support enable me to give today a contribution to the field.

I am very grateful to François Murat and Luc Tartar for many discussions on mathematical analysis these last 20 years, which were very helpful in the analysis of the turbulence models, since my first paper about the  $k - \mathcal{E}$  model in 1994 up to this point.

A particular thank goes to Luigi Berselli for fruitful discussions on the analysis part of the book. I am also grateful to Christian Leonard for our discussions on probability theory.

I address very special and warm thanks to Kevin Dunseath, who did a great work in proofreading and correcting my mistakes in English.

I thank Jean-Michel Coron, Li Tatsien, and the French-Chinese Mathematical Institute, ISFMA, for their support, and the hospitality of Fudan University, Shanghai (China), during the spring of 2012 and the summer of 2013, where parts of this book have been thought and written.

I am grateful to the Departamento de Ecuaciones Diferenciales y Análisis Numérico of the University of Seville (Spain) for its kind welcome every year since 1994.

I thank the Institute of Mathematical Research of Rennes (IRMAR), the University of Rennes 1, and INRIA of Rennes (France) for their support and in particular my colleagues of the mechanical team of IRMAR: Isabelle Gruais, Loic Le Marrec, Lalao Rakotomanana, Fulgence Razafimahery, and Nathalie Ritemard. I also thank the administrative and IT staff of IRMAR who helped me in many occasions, in particular Claude Boschet, Olivier Garo, Marie-Aude Verger, Patrick Pérez, and Carole Vosiak. I finally thank Etienne Memin and Fluminance at INRIA.

Thoughts go to Django Reinhardt, Wes Montgomery, B.B. King, Frank Zappa, Jimi Hendrix, Jimmy Page, and Slash, whose guitars have been close to me all the time.

Finally special thoughts and giant thanks go to my wife Sophie, who has always been by my side as I was working on this book for the last 3 years and to whom I owe a lot.

Rennes, France  
January 2014

Roger Lewandowski

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# Chapter 1

## Introduction

Understanding turbulence is one of the oldest and most challenging scientific problems. Since the early works of Boussinesq and Reynolds in the late nineteenth century that formalized the basic characteristics of turbulent flows, the analysis of the extremely complex behavior of turbulence has raised the interest of many researchers. The issue of what is “turbulence” is still far from being solved, although some facts can be deduced from observations and experiments. Turbulent flows have a huge impact in human life, from weather forecasting to freshwater supply, energy generation, navigation, biological processes and so on.

Numerical simulation of turbulence is thus of primary importance to improve human life in many ways. Classical fluid mechanics establishes that the motion of a viscous fluid is governed by the Navier–Stokes equations, which in theory should be appropriate to perform numerical simulations of turbulent flows.

However, a turbulent flow is a highly irregular system, characterized by chaotic property changes involving a wide range of scales in nonlinear interaction with each other. These features yield a high computational complexity, which makes today direct numerical simulations of turbulent flows from the Navier–Stokes equations impossible. This is why turbulence models are introduced, in order to reduce this computational complexity.

Besides experiments and physics, mathematical modeling and analysis play a central role in the study of turbulent flows. Mathematics provide a permanent support to build turbulent models for weather forecasting and meteorology, oceanography, climatology, and environmental and industrial applications. Industrial flow softwares (Ansys-Fluent, Comsol, Femap, ...) are deeply based upon mathematical and numerical analysis.