APPLIED EXTERIOR CALCULUS

DOMINIC G. B. EDELEN

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DOMINIC G. B. EDELEN
Center for the Application of Mathematics
Lehigh University

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PREFACE

The purpose of this book is to provide upper division undergraduate and beginning graduate students with access to the exterior calculus. Such access is essential simply because, much of the modern literature both in mathematics and in the quantified sciences has come to use the exterior calculus as an expository vehicle. Whole segments of the literature and, of greater importance, essential and often simplifying concepts are thus unintelligible to the student who is unfamiliar with the exterior calculus.

Most texts that treat the exterior calculus are oriented toward global results and the needs of the research worker. The reader thus is confronted at the very beginning, and rightly so, with fundamental aspects of topology and the theory of differentiable manifolds. The demands on the student are great, a quantum jump in both sophistication and maturity often being self-evident from the start. A significant portion of the exterior calculus and its attendant concepts can be mastered, however, without recourse to global concepts. In fact, most of the theory can be developed using only local notions, and it can be done in such a way that it places only moderate demands on a student with previous exposure to upper division algebra and analysis courses. These priorities are the basis on which this book has been written.

The analysis and discussions are confined from the outset to local questions. We start with standard n-dimensional number space and restrict attention to what happens in a single neighborhood of a point. It is even possible to restrict attention to what happens in a single neighborhood of a point that carries a single fixed coordinate cover. This, however, is adequate for discussion of tangents to curves and for the attachment of an n-dimensional vector space to each point of the neighborhood in a natural way. Tangent spaces and vector fields then follow as direct consequences, and a change of viewpoint leads to the realization of a vector field as a derivation of the associative algebra of C^∞ functions. Careful consideration of the properties of vector fields under mappings between neighborhoods of points in different spaces then give the more customary results. Exterior forms of degree 1 become immediately available as elements of the dual space of the tangent space. Forms of higher degree are defined operationally by introducing exterior products of basis elements for 1-forms. Again careful study of the mapping properties for exterior forms, which are induced by mappings between neighborhoods of points in different spaces, leads to the pull back map and to the standard results obtained in the more customary approach. With mastery of these facts, the exterior derivative and the Lie derivative follow in a natural way. The notions of closed and exact forms lead to the theory of linear homotopy operators that play the role of inverses of the exterior derivative on an appropriately defined subalgebra of the exterior algebra.

This approach places strong emphasis on the ability to compute at the earliest possible stage. Specific examples are given as soon as the student has been shown how to perform nontrivial calculations. This is further advanced by a list of straightforward but nontrivial exercises at the end of each of the first five chapters. The student is strongly encouraged to try as many of these exercises as possible and to compare answers with those that are given. If you cannot compute with facility, then there is something wrong, something missed along the way.

A second aim of the first five chapters is to provide the student with a sound foundation for modern studies in partial differential equations. While nineteenth-century mathematics was primarily concerned with developing specific methods to solve specific equations, it is twentieth-century mathematics that has developed general techniques for looking at large classes of problems. This is why exterior algebra, vector fields, differential forms, Lie derivatives, and group-theoretic methods were invented. Many of the exotic changes of variables that proliferate in the older literature as special tricks are both immediate and natural from the vantage point of the exterior calculus. Thus vector fields are used directly for a full discussion of the method of characteristics, whereby general solutions are constructed to linear and quasilinear partial differential equations of the first order. This is followed by study of systems of quasilinear partial differential equations with the same principal part. Systems of simultaneous linear partial differential equations are then studied by means of Lie subalgebras and reduction to Jacobi normal form. Partial differential equations are revisited in the context of exterior forms by a systematic exploitation of the Frobenius, Darboux, and Cartan theorems for differential systems. Antiexact forms and the linear homotopy operator, which are the subjects of Chapter Five, have their primary practical significance in answering the question of how to go about solving exterior differential equations in a systematic manner. This leads directly to the study of connection, tension, and curvature forms and to the Cartan equations of structure.

The student with a more applied flair may be put off by the definition, theorem, and proof format adopted throughout the first seven chapters. It is my personal preference because it causes the key ideas to stand out in bold relief and provides a means of rapid perusal and review. It is essential that the student learn to articulate specific points and established facts in a careful and exact manner, for without this ability things rapidly degenerate to utter confusion. Once the student has become accustomed to this mode of exposition, it is often a revelation how a careful statement and summary can point to new and previously unsuspected directions of investigation.

This is a book on applied exterior calculus, as the title states. The text has therefore been divided into three parts. Part One, comprising the first five chapters, covers the essential elements of the exterior calculus together with a number of abbreviated applications. Chapters Six and Seven, which comprise Part Two, deal with specific detailed applications of the exterior calculus to group-theoretic questions in nonlinear second-order partial differential equations and to problems in the calculus of variations. These two chapters are reasonably complete and bring the student to the frontier of modern work. They also provide background material necessary for understanding what follows.

Part Three consists of three chapters that provide in-depth studies of physical disciplines *via* the exterior calculus. Here we relax the definition, theorem, proof format in favor of direct exposition. Chapter Eight is concerned with classical and irreversible thermodynamics. It is based largely on Carathéodory's approach (the Darboux theorem and inaccessibility) with nontrivial extensions to nonconservative mechanical forces and internal degrees of freedom. The fundamental problem of irreversible thermodynamics is then stated and solved through use of the linear homotopy operator introduced in Chapter Five.

Chapter Nine studies electrodynamics with both electric and magnetic charges. General solutions of Maxwell's equations are constructed by converting the field equations to an equivalent system of exterior differential equations, followed by integration through use of the linear homotopy operator. The solutions are obtained without recourse to constitutive relations and are shown to give rise to a four-parameter group of general duality transformations. Restrictions imposed by the vacuum ether relations are then studied. Variational principles for electromagnetic fields in the presence of magnetic charge are obtained through standard techniques of the exterior calculus. This gives rise to closed form evaluations of the 1-forms of forces that act on electric and on magnetic charge distributions that include radiation reaction.

Chapter Ten deals with the modern theory of gauge fields. Here the full scope of the exterior calculus and much of the material in Chapters Six and Seven combine in what is now considered one of the cornerstones in the conceptual containment of physical reality. The simpler situation of a matrix Lie group of internal symmetries is considered first. Derivation of all relevant field equations and geometric structures are obtained even though no direct use is made of the theory of fiber bundles. This is followed by a general theory of operator-valued connections where the group action is allowed to be nonlinear and to act indiscriminately both on the physical state variables and on the space-time labels. Again, all relevant field equations and geometric structures are worked out, but this time specific application is made to gauging the Poincaré group.

Readers who are familiar with the exterior calculus may wish to proceed rapidly to the applications. To make this as simple and painless as possible, the following plan of abbreviated review is offered. Chapters Six through Ten listed below are followed by sequences of chapters. The boldface number refers to the chapter to be reviewed while the numbers to its right refer to the sections. It will often be necessary to read only the main theorems of the sections cited in order to proceed.

Chapter 6.	**2:**	1, 2, 3, 4, 6, 7,
	3:	1, 2, 4, 5, 6, 7,
	4:	1, 2, 5, 6, 7.
Chapter 7.	**2:**	1, 2, 3, 4, 6, 7,
	3:	1, 2, 4, 5, 6, 7,
	4:	1, 2, 5, 6, 7,
	6:	1, 2, 3, 4.
Chapter 8.	**2:**	1, 2, 3, 4, 6,
	3:	1, 2, 4, 5, 7,
	4:	1, 2, 4,
	5:	1, 2, 3, 4, 5.
Chapter 9.	**2:**	1, 2, 3, 4, 8,
	3:	1, 2, 4, 5,
	4:	1, 2, 4, 7,
	5:	1, 2, 3, 4, 5, 18, 19,
	7:	1, 2, 3.
Chapter 10.	**2:**	1, 2, 3, 4, 6, 7,
	3:	1, 2, 4, 5, 7,
	4:	1, 2, 5, 7,
	5:	1, 2, 3, 4, 5, 11, 15,
	6:	1, 2, 3, 4,
	7:	1, 2, 4,
	9:	1, 2, 3, 7.

This book is the outgrowth of a course on applied exterior calculus given at Lehigh University over the last eight years and owes much of its final form and substance to the many students who have helped remarkably. I am also indebted to the publishing house of Sijthoff and Noordhoff for their kind permission to use certain materials from the appendix of a previous book (*Isovector Methods for Equations of Balance*); in particular, Chapter Five on antiexact differential forms and linear homotopy operators. The labor in preparation of the many drafts of the manuscript was significantly lightened by the able secretarial assistance of Mary Connell and Lisa Ziegler. Finally, I wish to express my heartfelt thanks to Demetrios Lagoudas for his unstinting assistance in proofreading.

D. G. B. EDELEN

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PART THREE. APPLICATIONS TO PHYSICS

PART ONE

VECTORS AND FORMS

CHAPTER ONE

MATHEMATICAL PRELIMINARIES

1-1. NUMBER SPACE, POINTS, AND COORDINATES

The simplicity and intrinsic beauty of any mathematical discipline rests on the fact that it starts with the familiar or self-evident and builds an edifice of often unexpected logical implications. Whether the logical implications are viewed as an end in themselves, or as a structured stage for the play of physical reality, due care must be taken in the articulation of the underlying precepts. For the purposes of these discussions, the starting point is taken to be n-dimensional number space, E_n, together with mappings between E_n and the real line $\mathbb{R}$. Since E_n is the n-fold Cartesian product of the real line, $\mathbb{R}$, it may be given a system of Cartesian coordinates, in which case we say that E_n is referred to a Cartesian coordinate cover. Great care must be exercised at exactly this point, for the reader must not assume that the referral of E_n to a Cartesian coordinate cover carries along with it the ability to measure distances between points in E_n or to measure angles between intersecting lines in E_n. The fundamental concept here is that of a continuous swarm of points and that we have labeled the points of this swarm by a Cartesian coordinate cover as a matter of expediency and convenience. Once the Cartesian coordinate cover is in place, we can use it to introduce a topology and thereby discuss notions of nearness, continuity, and convergence. In particular, we shall say that a set in E_n is open if it is open in the Euclidean topology introduced on E_n by referral to a given Cartesian coordinate cover. For the time being, once E_n is referred to a Cartesian coordinate cover, this coordinate cover will remain fixed. What happens when the coordinate cover is changed will be taken up later.

Let U be an open set of E_n and let P be a generic point in U. The Cartesian coordinate cover of E_n can be used to assign any point P in U its coordinates $x^1(P), x^2(P), \ldots, x^n(P)$. This is conveniently written $P:(x^1, \ldots, x^n)$, or simply $P:(x^i)$ where it is understood that i runs from 1 through n. If f denotes a function that is defined on E_n and takes its values on the real line $\mathbb{R}$, we write

$$f(x^i) \quad \text{for } f(x^1, x^2, \ldots, x^n)$$

and sometimes even $f(x)$ when it is clear that x stands for $(x^i) = (x^1(P), x^2(P), \ldots, x^n(P))$. The function $f(x^i)$ then serves to define a map F from E_n into $\mathbb{R}$. We denote this situation by

$$F: E_n \to \mathbb{R} \,|\, t = f(x^i),$$

where the equation following the vertical slash gives a realization of the map F.

Suppose that we are given n real valued functions $\phi^1(t), \phi^2(t), \ldots, \phi^n(t)$ of the real variable t, where each of these functions is defined on some common open set $J \subset \mathbb{R}$. If we set $x^1 = \phi^1(t)$, $x^2 = \phi^2(t), \ldots, x^n = \phi^n(t)$, we obtain a curve in E_n; namely, a map Φ from the set J contained in the real line, $\mathbb{R}$, with coordinate t into the n-dimensional number space E_n. In this instance, we write

$$\Phi: J \subset \mathbb{R} \to E_n | x^i = \phi^i(t), \qquad i = 1, \ldots, n,$$

where the equations following the vertical slash give a realization of the map Φ. Simplification obtains if we write

$$\Phi: \mathbb{R} \to E_n | x^i = \phi^i(t),$$

where it is understood that the set $J \subset \mathbb{R}$, on which all n of the functions $\{\phi^i(t)\}$ are defined, is implied and that i runs from 1 through n.

There will be many instances in which we will have to write quantities such as

$$\sum_{i=1}^{n} g^i(x^k) W_i(x^k).$$

This can be significantly simplified by adopting the Einstein summation convention: *if an upstairs and a downstairs index are the same letter, then summation over the range of the index is implied*. Thus,

$$g^i(x^k) W_i(x^k) \equiv \sum_{i=1}^{n} g^i(x^k) W_i(x^k)$$

and

$$P^{ij}_{kr} H^k_j \equiv \sum_{j=1}^{n} \sum_{k=1}^{n} P^{ij}_{kr} H^k_j = U^i_r.$$