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Knot Invariants and Higher Representation Theory

Ben Webster

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Abstract

We construct knot invariants categorifying the quantum knot variants for all representations of quantum groups. We show that these invariants coincide with previous invariants defined by Khovanov for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ and by Mazorchuk-Stroppel and Sussan for $\mathfrak{sl}_n$.

Our technique is to study 2-representations of 2-quantum groups (in the sense of Rouquier and Khovanov-Lauda) categorifying tensor products of irreducible representations. These are the representation categories of certain finite dimensional algebras with an explicit diagrammatic presentation, generalizing the cyclotomic quotient of the KLR algebra. When the Lie algebra under consideration is $\mathfrak{sl}_n$, we show that these categories agree with certain subcategories of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_k$.

We also investigate the finer structure of these categories: they are standardly stratified and satisfy a double centralizer property with respect to their self-dual modules. The standard modules of the stratification play an important role as test objects for functors, as Vermas do in more classical representation theory.

The existence of these representations has consequences for the structure of previously studied categorifications. It allows us to prove the non-degeneracy of Khovanov and Lauda's 2-category (that its Hom spaces have the expected dimension) in all symmetrizable types, and that the cyclotomic quiver Hecke algebras are symmetric Frobenius.

In work of Reshetikhin and Turaev, the braiding and (co)evaluation maps between representations of quantum groups are used to define polynomial knot invariants. We show that the categorifications of tensor products are related by functors categorifying these maps, which allow the construction of bigraded knot homologies whose graded Euler characteristics are the original polynomial knot invariants.

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CHAPTER 1

Introduction

1. Quantum topology

Much of the theory of quantum topology rests on the structure of monoidal categories and their use in a variety of topological constructions. In this paper, we define a categorification of one of these constructions: the R-matrix construction of quantum knot invariants, inspired by the work of Reshetikhin and Turaev [**Tur88**, **RT90**].

Their work is in the context of the tensor category of representations of a quantized universal enveloping algebra $U_q(\mathfrak{g})$. They assign natural maps between tensor products to each ribbon tangle labeled with representations. One special case of this is a polynomial invariant of framed knots for each finite-dimensional representation of $U_q(\mathfrak{g})$. These maps are natural with respect to tangle composition; thus they can be reconstructed from a small number of constituents: the maps associated to a ribbon twist, crossing, cup and cap. The map associated to a link whose components are labeled with a representation of $\mathfrak{g}$ is thus simply a Laurent polynomial.

Particular cases of these invariants include:

- the **Jones polynomial** when $\mathfrak{g} = \mathfrak{sl}_2$ and all strands are labeled with the defining representation.
- the **colored Jones polynomials** for other representations of $\mathfrak{g} = \mathfrak{sl}_2$.
- specializations of the **HOMFLYPT polynomial** for the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$.
- the **Kauffman polynomial** (not to be confused with the Kauffman bracket, a variant of the Jones polynomial) for the defining representation of $\mathfrak{so}_n$.

These special cases have been categorified to knot homologies from a number of perspectives, beginning with work of Khovanov on the Jones polynomial. However, the vast majority of representations previously had no homology theory attached to them. In this paper, we will construct such a theory for any labels; that is:

Theorem A.1 *For each simple complex Lie algebra $\mathfrak{g}$, there is a homology theory $\mathcal{K}(L, \{\lambda_i\})$ for links L whose components are labeled by finite dimensional representations of $\mathfrak{g}$ (here indicated by their highest weights λ_i), which associates to such a link a bigraded vector space whose graded Euler characteristic is the quantum invariant of this labeled link.*

Given the extensive past work on knot homology, it is natural to ask which of the homology theories mentioned above coincide with those of Theorem A.1 in special cases.

Theorem A.2 *When $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$ and the link is labeled with the defining representation of these algebras, the theory $\mathcal{K}(L, \{\lambda_i\})$ coincides up to grading shift with Khovanov's homologies for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$. In the case $\mathfrak{g} = \mathfrak{sl}_n$ and we use the defining representation, $\mathcal{K}(L, \{\lambda_i\})$ agrees with the Mazorchuk-Stroppel-Sussan homology.*

Previous approaches to categorifying the special Reshetikhin-Turaev invariants mentioned above have been given by Khovanov and Khovanov-Rozansky [**Kho00**, **Kho02**, **Kho04**, **Kho07**, **KR08b**, **KR07**, **KR08a**], Stroppel and Mazorchuk-Stroppel [**Str05**, **MS09**], Sussan [**Sus07**], Seidel-Smith [**SS06**], Manolescu [**Man07**], Cautis-Kamnitzer [**CK08a**, **CK08b**], Mackaay, Stošić and Vaz [**MSV09**, **MSV11**] and the author and Williamson [**WW**]. All of these approaches depend heavily on special features of minuscule representations.

There has been some progress on other representations of $\mathfrak{sl}_2$. In a paper still in preparation, Stroppel and Sussan also consider the case of the colored Jones polynomial [**SS**] (building on previous work with Frenkel [**FSS12**]); it seems likely their construction is equivalent to ours (see Chapter 4). Similarly, Cooper and Krushkal have given a categorification of the colored Jones polynomial using Bar-Natan's cobordism formalism for Khovanov homology [**CK12**]. We show in [**Webf**] that Cooper and Krushkal's theory agrees with ours for colored Jones polynomials.

On the other hand, the work of physicists suggests that categorifications for all representations exist; one schema for defining them is given by Witten [**Wit**]. The relationship between Witten's proposals and the invariants presented in this paper is completely unknown (at least to the author) and presents a very interesting question for consideration in the future.

Another question of particular interest is whether $\mathcal{K}(L, \{\lambda_i\})$ for the defining representation of $\mathfrak{sl}_n$ coincides with Khovanov-Rozansky homology [**KR08b**]; we will establish this agreement in future work with Mackaay [**MW**].

At the moment, we have not proven that $\mathcal{K}(L, \{\lambda_i\})$ is functorial, but we do have a proposal for the map associated to a cobordism when the weights λ_i are all minuscule. As usual in knot homology, this proposed functoriality map is constructed by picking a Morse function on the cobordism, and associating simple maps to the addition of handles. We have no proof that this definition is independent of Morse function and we anticipate that proving this will be quite difficult.

2. Categorification of tensor products

The program of "higher representation theory," begun (at least as an explicit program) by Chuang and Rouquier in [**CR08**] and continued by Rouquier [**Roua**] and Khovanov-Lauda [**KL10**], is aimed at studying "2-analogues" of the universal enveloping algebras of simple Lie algebras $U(\mathfrak{g})$ and their quantizations $U_q(\mathfrak{g})$. The 2-analogue for us of the quantum group $U_q(\mathfrak{g})$ is the strict 2-category $\mathcal{U}$ defined in [**CL15**, §2]. In this paper, we'll define an algebra $T^{\underline{\boldsymbol{\lambda}}}$ for each list $\underline{\boldsymbol{\lambda}} = (\lambda_1, \ldots, \lambda_\ell)$ of dominant weights for any symmetrizable Kac-Moody algebra $\mathfrak{g}$. Our 2-analogue of a tensor product of simple $\mathfrak{g}$-representations is the category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ of finite-dimensional representations of $T^{\underline{\boldsymbol{\lambda}}}$.

Our first objective is to show that we have defined a categorification of such tensor products.

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Theorem B *The category $\mathfrak{V}^{\underline{\lambda}}$ carries a categorical action of $\mathfrak{g}$, that is, it carries an action of the 2-category $\mathcal{U}$. The Grothendieck group of $\mathfrak{V}^{\underline{\lambda}}$ is canonically isomorphic to the tensor product*

$$V_{\underline{\lambda}} \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$$

of irreducible representations V_{λ_i}.

In the case where $\mathfrak{g}$ is finite-type and simply-laced, the classes of indecomposable projectives in $\mathfrak{V}^{\underline{\lambda}}$ are Lusztig's canonical basis of a tensor product by [**Web15**, 6.11].

When $\underline{\lambda} = (\lambda)$, the algebra $T^{\lambda} := T^{(\lambda)}$ is a cyclotomic KLR algebra in the sense of [**KL10**, §3.4]. Even in this case, the categorical action of Theorem B is new, and it implies that the induction and restriction functors on these categories are biadjoint. This was proved independently by Kang and Kashiwara [**KK12**] using completely different methods. This action can be used to prove that the 2-category $\mathcal{U}$ is nondegenerate in the finite type case. Earlier versions of this paper included a discussion of non-degeneracy outside of finite type, which is now proven in [**Webb**] instead.

These algebras $T^{\underline{\lambda}}$ are also quite interesting from the perspective of pure representation theory. We'll prove that they have a number of properties which have already appeared in the literature.

Theorem C *The projective-injective objects of $\mathfrak{V}^{\underline{\lambda}}$ form a categorification of the subrepresentation $V_{\lambda_1+\cdots+\lambda_n} \subset V_{\underline{\lambda}}$. In particular, if $\underline{\lambda} = (\lambda)$, then all projectives are injective and the algebra $T^{(\lambda)}$ is Frobenius.*

The sum of all indecomposable projective-injectives has the double centralizer property. This realizes $T^{\underline{\lambda}}$ as the endomorphisms of a natural collection of modules over the algebra $T^{(\lambda_1+\cdots+\lambda_n)}$.

The algebra $T^{\underline{\lambda}}$ is standardly stratified. The semi-orthogonal decomposition for this stratification categorifies the decomposition of $V_{\underline{\lambda}}$ as the sum of tensor products of weight spaces.

These algebras also have connections in the type A case to classical representation theory, as has been explored by Brundan and Kleshchev [**BK09**]. Using Theorem C, we will build on their work in Chapter 9 by showing that the algebras $T^{\underline{\lambda}}$ are endomorphism algebras of certain projectives in parabolic category $\mathcal{O}$, while in type $\widehat{A}$, they are related to the representations of the cyclotomic q-Schur algebra. This last relationship will be explored more fully in work of the author and Stroppel [**SW**, **Webe**].

The method of proof leaves little hope for finding connections with category $\mathcal{O}$ in other types. We see no reason to think that $\mathfrak{V}^{\underline{\lambda}}$ has a similar description in terms of classical representation theory when $\mathfrak{g} \ncong \mathfrak{sl}_n, \widehat{\mathfrak{sl}}_n$, though we would be quite pleased to be proven wrong in this speculation.

3. Topology

We now turn to the construction of knot invariants. As mentioned above, the original construction of Reshetikhin-Turaev invariants is encoded in a ribbon structure on the category of $U_q(\mathfrak{g})$-representations. We can depict the structure maps of this category in terms of diagrams.

- Crossing two ribbons: the corresponding operator in representations of the quantum group is called the **braiding** or **R-matrix**[1]. More generally for any braid σ on ℓ strands, this defines a homomorphism $\Phi(\sigma)\colon V^{\underline{\lambda}} \to V^{\sigma(\underline{\lambda})}$.
- Creating a cup, or closing a cap: the corresponding operators in representations of the quantum group are called the **(co)evaluation** and **quantum (co)trace**.
- Adding a full twist to one of the ribbons: the corresponding operator in the quantum group is called the **ribbon element**.

In this paper, we will categorify every of these maps to a functor, and use these the define the invariants $\mathcal{K}(L, \{\lambda_i\})$. This approach was pioneered by Stroppel for the defining rep of $\mathfrak{sl}_2$ [**Str09**, **Str**] and was extended to $\mathfrak{sl}_n$ by Sussan [**Sus07**] and Mazorchuk-Stroppel [**MS09**]. To work in complete generality, we must use the derived categories[2] $\mathcal{V}^{\underline{\lambda}} = D(\mathfrak{V}^{\underline{\lambda}})$ of finite dimensional $T^{\underline{\lambda}}$-representations, rather than the variations of category $\mathcal{O}$ used by those authors.

Theorem D *The derived category $\mathcal{V}^{\underline{\lambda}}$ carries functors categorifying all the structure maps of the ribbon category of $U_q(\mathfrak{g})$-modules:*

(i) If σ is a braid, then we have an exact functor $\mathbb{B}_\sigma\colon \mathcal{V}^{\underline{\lambda}} \to \mathcal{V}^{\sigma(\underline{\lambda})}$ such that the induced map on Grothendieck groups $K_0(T^{\underline{\lambda}}) \to K_0(T^{\sigma(\underline{\lambda})})$ coincides with $\Phi(\sigma)$. Furthermore, these functors induce a strong action of the braid groupoid on the categories associated to permutations of the set $\underline{\lambda}$.

(ii) If two consecutive elements of $\underline{\lambda}$ label dual representations and $\underline{\lambda}^-$ denotes the sequence with these removed, then there are functors $\mathbb{T}, \mathbb{E}\colon \mathcal{V}^{\underline{\lambda}} \to \mathcal{V}^{\underline{\lambda}^-}$ which induce the quantum trace and evaluation on the Grothendieck group, and similarly functors $\mathbb{K}, \mathbb{C}\colon \mathcal{V}^{\underline{\lambda}^-} \to \mathcal{V}^{\underline{\lambda}}$ for the coevaluation maps and quantum cotrace maps.

(iii) When $\mathfrak{g} = \mathfrak{sl}_n$, the structure functors above can be described in terms of twisting and Enright-Shelton functors on $\mathcal{O}$.

By definition (see [**CP95**, §4]), the quantum knot invariants are given by a composition of the decategorifications of the functors constructed in Theorem D. Combining the functors themselves in the same pattern gives the knot homology of Theorem A.1.

4. Summary

Let us now summarize the structure of the paper.

- In Chapters 2 and 3, we discuss the basics of the 2-category $\mathcal{U}$, and prove it acts on $\mathfrak{V}^\lambda$. This is accomplished by the construction of categorifications $\mathcal{U}_i^-$ for the minimal non-solvable parabolics $U(\mathfrak{p}_i)$. These categories carry a mixture of the characteristics of $U(\mathfrak{b})$ and $U(\mathfrak{sl}_2)$; an appropriate non-degeneracy result is already known for both of these algebras separately.

[1] As usual, the R-matrix is a map between tensor products of representations $V \otimes W \to V \otimes W$ intertwining the usual and opposite coproducts; we use the term **braiding** to refer to the composition of this with the usual flip map, which is thus a homomorphism of representations $V \otimes W \to W \otimes V$.

[2] We'll want to use slightly unusual finiteness conditions on $D(\mathfrak{V}^{\underline{\lambda}})$, so we'll leave the precise definition of these categories to the body of the paper. See Definition 4.8.

By modifying the proofs of these previous results, we can show that $\mathcal{U}_i^-$ acts on $\mathfrak{V}^\lambda$. It is an easy consequence of this that the full $\mathcal{U}$ acts. These results are of independent interest (and, in fact, some of them have been proven independently by Kang and Kashiwara [**KK12**]).

- In Chapter 4, we define the algebras $T^{\underline{\boldsymbol{\lambda}}}$, using the familiar tool of graphical calculus. This graphical calculus gives an easy description of the action of the category $\mathcal{U}$. We also study the relationship of this category to $T^{(\lambda_1+\cdots+\lambda_\ell)}$.
- In Chapter 5, we develop a special class of modules which we term **standard modules**, which define a standardly stratified structure. In the case where all λ_i are minuscule, this structure is even quasi-hereditary. These modules also serve as categorifications of pure tensors.
- In Chapter 6, we prove Theorem D(i). That is, we construct the functor lifting the braiding of the monoidal category of $U_q(\mathfrak{g})$-representations. This functor is the derived tensor product with a natural bimodule. Particularly interesting and important special cases correspond to the half-twist braid, which sends projective modules to tiltings, and the full twist braid, which gives the right Serre functor of $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$.
- In Chapter 7, we prove Theorem D(ii). The most important element of this is to identify a special simple module in the category for a pair of dual fundamental weights, which categorifies an invariant vector.
- In Chapter 8, we prove Theorem A.1 using the functors constructed in Theorem D and a small number of explicit computations. We also suggest a map for the functoriality along a cobordism between links in the minuscule case. As mentioned before, it is unknown whether this is independent of choices.
- In Chapter 9, we consider the case $\mathfrak{g} = \mathfrak{sl}_n$ or $\widehat{\mathfrak{sl}}_n$. In this case, we employ results of Brundan and Kleshchev to show that $T^{\underline{\boldsymbol{\lambda}}}$ is in fact the endomorphism algebra of a projective in a parabolic category $\mathcal{O}$ in finite type. In affine type, there is a similar description using the cyclotomic q-Schur algebra. In Chapter 4, we relate the functors appearing in Theorem D to previously defined functors on category $\mathcal{O}$. This allows us to show the portions of Theorem A.1 regarding comparisons to Khovanov homology and Mazorchuk-Stroppel-Sussan homology.

We should note that an earlier version of this paper contained a section on the connection between the algebraic material in this paper to the geometry of quiver varieties and canonical bases. In the interest of length and heaviness of machinery, that material has been moved to other papers [**Weba**, **Webg**, **Web15**].

Notation

We let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, which we will assume is fixed for the remainder of the paper. Let Γ denote the Dynkin diagram of this algebra, considered as an unoriented graph. We'll also consider the weight lattice $Y(\mathfrak{g})$, root lattice $X(\mathfrak{g})$, simple roots α_i and coroots $\alpha_i^\vee$. Let $c_{ij} = \alpha_i^\vee(\alpha_j)$ be the entries of the Cartan matrix.

We let $\langle -,-\rangle$ denote the symmetrized inner product on $Y(\mathfrak{g})$, fixed by the fact that the shortest root has length $\sqrt{2}$ and

$$2\frac{\langle \alpha_i, \lambda\rangle}{\langle \alpha_i, \alpha_i\rangle} = \alpha_i^\vee(\lambda).$$

As usual, we let $2d_i = \langle \alpha_i, \alpha_i\rangle$, and for $\lambda \in Y(\mathfrak{g})$, we let

$$\lambda^i = \alpha_i^\vee(\lambda) = \langle \alpha_i, \lambda\rangle / d_i.$$

We note that we have $d_i c_{ij} = d_j c_{ji} = \langle \alpha_i, \alpha_j\rangle$ for all i, j.

We let ρ be the unique weight such that $\alpha_i^\vee(\rho) = 1$ for all i and $\rho^\vee$ the unique coweight such that $\rho^\vee(\alpha_i) = 1$ for all i. We note that since $\rho \in {}^1\!/_2 X$ and $\rho^\vee \in {}^1\!/_2 Y^*$, for any weight λ, the numbers $\langle \lambda, \rho\rangle$ and $\rho^\vee(\lambda)$ are not necessarily integers, but $2\langle \lambda, \rho\rangle$ and $2\rho^\vee(\lambda)$ are (not necessarily even) integers.

Throughout the paper, we will use $\underline{\boldsymbol{\lambda}} = (\lambda_1, \ldots, \lambda_\ell)$ to denote an ordered ℓ-tuple of dominant weights, and always use the notation $\lambda = \sum_i \lambda_i$.

We let $U_q(\mathfrak{g})$ denote the deformed universal enveloping algebra of $\mathfrak{g}$; that is, the associative $\mathbb{C}(q)$-algebra given by generators E_i, F_i, K_μ for $i \in \Gamma$ and $\mu \in Y(\mathfrak{g})$, subject to the relations:

i) $K_0 = 1$, $K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in Y(\mathfrak{g})$,
ii) $K_\mu E_i = q^{\alpha_i^\vee(\mu)} E_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,
iii) $K_\mu F_i = q^{-\alpha_i^\vee(\mu)} F_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,
iv) $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{q^{d_i} - q^{-d_i}}$, where $\tilde{K}_{\pm i} = K_{\pm d_i \alpha_i}$,
v) For all $i \neq j$

$$\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \qquad \text{and} \qquad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0.$$

This is a Hopf algebra with coproduct on Chevalley generators given by

$$\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i \qquad \Delta(F_i) = F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i$$

We let $U_q^{\mathbb{Z}}(\mathfrak{g})$ denote the Lusztig (divided powers) integral form generated over $\mathbb{Z}[q, q^{-1}]$ by $\frac{E_i^n}{[n]_q!}, \frac{F_i^n}{[n]_q!}$ for $n \geq 0$. We let $\dot{U}^{\mathbb{Z}}$ be the algebra obtained by adjoining idempotents 1_μ projecting to integral weight spaces to $U_q^{\mathbb{Z}}(\mathfrak{g})$. The integral form of the representation of highest weight λ over this quantum group will be denoted by $V_\lambda^{\mathbb{Z}}$. We will always think of this integral form as generated by a fixed highest weight vector v_λ. For a sequence $\underline{\boldsymbol{\lambda}}$, we will be interested in the tensor product

$$V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} = V_{\lambda_1}^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \cdots \otimes_{\mathbb{Z}[q,q^{-1}]} V_{\lambda_\ell}^{\mathbb{Z}}.$$

The category of representations over $U_q(\mathfrak{g})$ is a braided monoidal category; of particular importance for us is the R-matrix. We use the opposite R_{21} of the R-matrix defined by Tingley in [**Tin**]; since Tingley uses the opposite coproduct, this will be an R-matrix for us. Thus, for two representations M, M', we have an isomorphism $\sigma_{M,M'} \colon M \otimes M' \to M' \otimes M$ sending $m \otimes m' \mapsto s(R_{12} m \otimes m')$. In the notation of [**Tin**], this map is defined by

$$\mathcal{R}_{M,M'}(m \otimes m') \in q^{\langle \mathrm{wt}(m), \mathrm{wt}(m')\rangle} m' \otimes m + \sum q^{\langle \mathrm{wt}(m)-\beta, \mathrm{wt}(m')+\beta\rangle} X_\beta m' \otimes Y_\beta m \tag{1.1}$$

where X_β has weight $\beta > 0$ and Y_β weight $-\beta > 0$. In particular, if m is lowest weight or m' highest weight, then this equation simplifies to $\mathcal{R}_{M,M'}(m \otimes m') \in q^{\langle \mathrm{wt}(m), \mathrm{wt}(m')\rangle} m' \otimes m$.

For not especially important technical reasons, it will also be helpful to consider $V_{\underline{\lambda}}^{1/D} = V_{\underline{\lambda}}^{\mathbb{Z}}[q^{1/D}]$ and $V_{\underline{\lambda}}^{\mathbb{C}} \cong V_{\underline{\lambda}}^{\mathbb{Z}}[\{q^z\}_{z\in\mathbb{C}}]$, this module with either the Dth roots of q (for a fixed D) or all complex powers of q adjoined. We will also consider the completion of these modules in the q-adic topology $V_{\underline{\lambda}} = V_{\underline{\lambda}}^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}((q))$.

For a graded ring R, we let R-mod denote the category of finitely generated right graded R-modules. For a graded R-module M, we let M_n denote the vectors of degree n. Let $K^0(R)$ denote the Grothendieck group of finitely generated graded projective right R-modules. This group carries an action of $\mathbb{Z}[q, q^{-1}]$ by grading shift $[A(i)] = q^i[A]$, where $A(i)_n = A_{i+n}$. The careful reader should note that this is opposite to the grading convention of Khovanov and Lauda.

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