SET THEORY
An Introduction

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PREFACE

By its nature, set theory does not depend on any previous mathematical knowledge. Hence, an individual wanting to read this book can best find out if he is ready to do so by trying to read the first ten or twenty pages of Chapter 1. As a textbook, the book can serve for a course at the junior or senior level. If a course covers only some of the chapters, the author hopes that the student will read the rest himself in the next year or two. Set theory has always been a subject which people find pleasant to study at least partly by themselves.

Chapters 1-7, or perhaps 1-8, present the core of the subject. (Chapter 8 is a short, easy discussion of the axiom of regularity). Even a hurried course should try to cover most of this core (of which more is said below). Chapter 9 presents the logic needed for a fully axiomatic set theory and especially for independence or consistency results. Chapter 10 gives von Neumann's proof of the relative consistency of the regularity axiom and three similar related results. Von Neumann's 'inner model' proof is easy to grasp and yet it prepares one for the famous and more difficult work of Gödel and Cohen, which are the main topics of any book or course in set theory at the next level. Chapter 9 might be slightly easier for someone who has already studied logic, but it is written to be understandable by a reader with no background in logic. Actually, some of the logic given in Chapter 9 is not covered in most first year logic courses (and most of what they do cover is not needed in Chapter 9 or 10). After Chapters 1-8, the thing most required for further work in set theory (and so the thing next to be included in a longer course) is the material of Chapter 9 (and Chapter 10). The last Chapter, 11, returns to 'straight' set theory and can be read after Chapter 7. Its first part adds to the earlier cardinal arithmetic, and its second part to the earlier ordinal arithmetic. Most people find these topics attractive and easy, and will indeed read them by themselves if a course does not cover them.

For many years, the widely used introductory books on set theory all presented intuitive set theory. For the past two or three decades, the exact opposite has been true: all such books have given axiomatic set theory. But for the student, the trivial and irritating business of fooling around, as he begins to learn set theory, with axioms (saying for example that $\{x,y\}$ exists!) discourages him from grasping the main, beautiful facts about infinite unions, cardinals, etc., which should be a joy.

Therefore, we shall work in intuitive set theory in the first five of the seven main chapters. The axioms are discussed in the very short Chapter 6. By that time, many of the special features of the axiomatic business will be seen by the student to be trivial, as they should be. At the end of Chapter 6 the reader has all of Chapters 1-5 behind him *axiomatically*. In Chapter 7 (on well-orderings) we now work from the axioms, but the reader sees at once that there is practically no difference between working intuitively and working axiomatically.

Two other pedagogical devices are used to increase the reader's speed in getting the main ideas – the first (which the author learned from Azriel Levy when he was teaching in Berkeley) is this: Cardinals, order types, etc., are not defined (in some ad hoc way) until Chapter 7; but, in Chapter 2, we just 'grab' them, as Cantor did. The other device imitates the famous book (or books) of Hausdorff [F1914 and F1927 (English edition 1957)] in putting off the serious study of well-orderings as long as possible – in fact until Chapter 7. (Even in Cantor's work, some ideas are less natural and easy then others!) As a side-effect, well-ordering is studied (in Chapter 7) while working axiomatically; and it is just possible that this subject (particularly definition by induction) is one of the few more easily grasped working axiomatically than intuitively.

The author is indebted to his own teacher in set theory and logic, Alfred Tarski, for many things in this book. Some recent students have suggested various shorter proofs, which have been gratefully used. The author is very grateful to Shaughan Lavine, who prepared the index and also assisted with the proofreading, making many corrections and improvements.

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INTRODUCTION

Set theory has two overlapping aspects. In one, it is a branch of mathematics, like algebra or differential geometry, with its own special subject matter. In its other aspect, set theory is not a branch of mathematics but the very root of mathematics from which all branches of mathematics rise. (In this picture only logic lies still below set theory. Together they are often called the 'foundations of mathematics.')

This Introduction contains some remarks about the (early) history of set theory. The book proper, the mathematics, begins with Chapter 1 and does not depend on the Introduction. The remarks below should be read (rapidly for pleasure) now or perhaps after reading much of the book, or both.

Set theory, in its aspect as one branch of mathematics, is devoted particularly to the theory of infinite cardinal and ordinal numbers. Perhaps not even relativity theory can be said to have sprung so completely from the mind of one man as did set theory (in this aspect)! That man was Georg Cantor. Cantor lived from 1845 to 1918, his main publications appearing between 1874 and 1897. (For reference to Cantor's papers see the bibliography of Fraenkel [1960].) Cantor was also one of the founders of point set topology which in turn had arisen in a study of trigonometric series. Cantor's set theory stands as one of the great creations of mathematics. David Hilbert is widely considered to be the leading mathematician of the last hundred years. Replying to some who thought that the paradoxes (see below) might destroy Cantor's theory of sets, Hilbert spoke of "a paradise created by Cantor from which nobody shall ever expel us" (cf. Fraenkel [1961], p. 240). Reader: Note what lies ahead for you!

The early story of the 'foundations aspect' of set theory goes back farther, and it is not concentrated in the work of one man. Actually, in the years 1800-1930 the entire mathematics went through a major change (at the very least, of *style*) into what every subject calls today its 'modern approach.' Set theory and logic played a key role in this change, but that role is nevertheless sometimes over-estimated. In fact, every branch of mathematics was going through the same convulsions. For example, in algebra the (central modern) notion of isomorphism began to appear by 1830 or earlier in the work of Galois and others, but the simple, general modern concept is not exactly present until perhaps 1910 (Steinitz on fields) or even 1920! One might think nothing special was happening here, as mathematics is always evolving. But B.L. von der Waerden's *Modern Algebra* [F1950], a graduate-level algebra book, appeared first in 1930, and is still widely used as a textbook today, fifty years later! During 1850-1930 the serious changes in style always going on would have made such a thing almost impossible. An essential feature of the modern approach is just that there is in a sense an *end of the line* in style and rigor. In most fields that was reached within ten years of 1930!

The early developments in (the foundations aspect of) set theory were closely connected with the 'convulsions' in *analysis* in the 19th century. In a reversal of the lack of rigor in the 18th century, the modern, rigorous approach to analysis began to appear in the early 1800's. By 1830, Cauchy and others were able to use *almost* the modern style in defining limits or continuity. One man ahead of his time went even further than Cauchy towards our modern analysis, namely the priest, B. Bolzano (1781-1848) (see [F1810], [F1837], and [F1851])*. Bolzano began, in particular, to use the notion of (arbitrary) *set* much more in analysis. In the same period the closely related notion of arbitrary *function* was emerging, e.g., in the work of Fourier (1810), Dirichlet (1840), and Riemann (1826-1866). (One date as in "Fourier (1810)," refers to the time of a key publication). Bolzano is indeed the only person ever proposed as a predecessor for Cantor (and was so even by Cantor himself). Bolzano had begun (but only begun) to study the notion 'A and B can be put in one-to-one correspondence' – the keystone of Cantor's theory.

By 1861, K. Weierstrass was able to give almost our idea of a 'modern' course in real variables! A decade or so later, Dedekind and Cantor created what are still the two main methods for constructing the reals from the rationals. This was a needed step in reducing all of mathematics to set theory. Richard Dedekind (1831-1916), an older mentor and longtime friend of Cantor's, contributed much to the set-theoretical study of the natural numbers. His famous book *Was sind und was sollen die Zahlen* [F1888] is full of passages which were new then but are now spoken in every beginning course in real variables or modern algebra. G. Peano (1895) is also famous for his contributions to the same subject.

Ernst Zermelo [F1904, 1908, 1908a, 1909, 1913] seems to have been the first to take each natural number to be a certain set. He also was the first to know (although in an odd way) how to get along with only $\in$ and without the ordered couple† (though later Norbert Wiener in 1915 pointed out the much more elegant

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*References like "Bolzano [F1851]" are explained at the beginning of the Bibliography.

†Roughly, Zermelo observed he could get along using only functions $f: A \to B$ where A and B are disjoint; and that such an f can be taken to be a suitable set of doubletons $\{a,b\}$. (The author learned this from Gregory Moore.)

fact that one can simply *define* a reasonable notion of ordered couple using only ϵ.) These two steps taken by Zermelo were among the last needed to show that one could do the entire mathematics in a set theory using only the notion 'belongs to.' Clearly these steps were not unrelated to Zermelo's most famous contribution, made in 1908, when he gave the first axiom system for set theory (and hence for all of mathematics). The most frequently used axioms today are nearly just his! Nevertheless, Zermelo's axiomatic work was lacking in one respect, as he did not realize that an axiomatic system cannot be fully understood until its underlying logic is fully understood.

Earlier, at exactly the same time as Cantor, there was a man, of perhaps the same brilliance as Cantor, who leaped ahead fifty years in the subject of *logic*. That man was Gottlob Frege. (His key publications were in 1879-1903). Some of his contributions will be mentioned a bit later on.

In the years 1897-1902, paradoxes (that is, contradictions) were discovered by Bertrand Russell and others using what perhaps *might* be taken as acceptable set-theoretical axioms. (Frege, almost alone, had actually assumed these axioms!) For many decades, the paradoxes were considered to be the central feature of set theory or at least of its foundations. For example, many books seemed to imply that the whole purpose of the axiomatic approach in set theory was to deal with the paradoxes (awkward for Euclid!) In recent times, the paradoxes have been assigned a lesser importance, though certainly a great one. In fact, Cantor himself knew that it was necessary to distinguish between 'ordinary' sets and very large, 'bad' collections. Obviously, Zermelo had very good reasons for studying the axiomatization of set theory (and the whole of mathematics!) – even without the paradoxes. But the paradoxes have certainly played a central role in mathematical philosophy for decades. Another controversy has been over the axiom of choice – which was made famous by Zermelo in 1904 when he derived from it the well-ordering principle. Both the paradoxes and the axiom of choice will be discussed (mathematically) in later chapters, where some brief remarks will be made on their history and, in general, on the history of set theory after 1900.

Let us return to the history of a fully correct axiomatization of set theory (which we shall follow up to 1930). Frege had provided just the understanding of logic and logistic systems needed for a fully correct axiomatization. (A logistic system is a deductive system in which the notion of what is a proof is so clear it can be decided by a machine.) Unfortunately, Frege's work was not widely known even by 1900. However, at almost the same time as Zermelo's axiomatization, another very different axiomatization of set theory was carried out by Bertrand Russell and Alfred North Whitehead in their famous three volume work, *Principia Mathematica*. From the point of view of a working mathematician, Zermelo's axioms were (and are) tremendously better than the awkward system of the *Principia*. However, Russell and Whitehead knew and appreciated Frege's work in logic. They included formal logic in their work and, although possibly with some flaws, their work was a logistic system (for set theory and the whole of mathematics). It is thus clear why this awkward work was so acclaimed and so influential.

There remained the (actually very easy) problem of getting a system both workable and fully logistic. In [F1923], Thoralf Skolem succeeded, say, 95% in simply making the working system of Zermelo into a logistic one. He proposed that Zermelo's 'schemas' (which caused the trouble) be replaced by the infinite set of their 'ϵ-instances' (exactly as is often done today). But, alas, it seems unlikely that Skolem grasped then the idea of a logistic system for logic. (In his other papers of the time, he always considers that the only meaning of 'logically valid' is 'holds in all models.') Nevertheless, it seems that in 1923, to the advanced logicians in the famous 'Hilbert group' in Germany and the famous 'Polish group,' Skolem's paper must have suggested the full logistic system, just as it does today.

In the very same paper, Skolem shared at least partly with Adolf Fraenkel [L1922] and Dmitri Mirimanoff [L1917] the (independent) discovery of the principal missing axiom in Zermelo's system, needed to carry out Cantor's original work – the axiom of replacement. (Fraenkel's influence in the matter was greatest and the axiom is often called 'Fraenkel's axiom.')

There is one other axiom usually included today, namely, the axiom of regularity. This axiom concerns 'partial universes' which are loosely related to Russell's 'types.' The partial universes and the axiom of regularity were first considered by Mirimanoff [L1917] (and also in Skolem [F1923]). They were given a definitive study by Johan (later John) von Neumann in a group of papers on set theory [F1923-1928].

In the same group of papers, the young von Neumann, who was to become perhaps the greatest mathematician of the first half of the twentieth century, published the first flawless axiomatization of set theory. In fact, it had its own awkwardness and has rarely been used! A workable modification of it was given by P. Bernays [F1937-1954]. Notice that the two systems of set theory (which do not differ very much) most used today (Zermelo-Fraenkel and von Neumann-Bernays) were achieving exactly their present form in just the years when van der Waerden's *Modern Algebra* appeared!

1. SETS AND RELATIONS AND OPERATIONS AMONG THEM

The mathematical content of Chapter 1 is easy; so the chapter should be read rather rapidly.

Before we begin to study sets, a brief remark about logic will be made. The reader is probably already familiar with at least some of the following logical symbols or abbreviations:

$\sim$ for *not, or it is not the case that.*
$\vee$ for *or.*
$\wedge$ for *and.*
_____ $\rightarrow$ -------- for *if* _____ *then* --------.
_____ $\leftrightarrow$ -------- for _____ *if and only if* --------. (Also: 'iff'.)
$\forall x$ _______ for *for all x,* _______.
$\exists x$ _______ for *for some x,* _______ or *there exists an x such that* _______.
$\exists ! x$ _______ for *there exists exactly one x such that* _______.
ιx _______ for *the unique x such that* _______.
$x = y$ for *x equals y,* or *x and y are identical.*

In theoretical discussions about the language we use – for example in Chapters 6 and 9 – it is very useful to suppose, or pretend, that we write in a purely symbolic language.

On the other hand, in actually doing mathematics it is desirable to write in pure English (or whatever ordinary language), not using any logical symbols (except $=$). The reason is that in the ordinary language one has a rich tradition of conventions for how to write grammatically and even for how to argue. On the other hand, in 'popular symbolic logic' there is no convention for dealing (as in a proof) with more than one sentence at a time. A *single sentence* in symbolic logic is readable, and hence logical symbols are occasionally used in the statement of a theorem or definition. If the reader will look at any current journal in mathematics (even on logic) he will find virtually no use of logical abbreviations!

Even though they will not be used much, the logical symbols listed above are helpful in understanding ordinary (English) mathematical usage. For example, we realize that 'for all x,' 'for any x,' 'for every x' and 'for each x' all are just the same (namely, '$\forall x$'). Moreover, our attention is called to the great importance in mathematics of the English passages above (corresponding to the symbols). In particular, most people need to overcome some odd initial reluctance to use themselves the phrase 'the unique x such that' – which one must do throughout modern mathematics!

§ 1. Set algebra and the set-builder

If we have conceived of some things (of any kind), then we can conceive of sets of these things – and sets of sets, etc., etc. We may have started with some things which are not sets themselves ('non-sets,' 'atoms,' or in German, 'Urelementen'). In one natural form of set theory we make just these assumptions, leaving undecided whether there are a small number or a large number of non-sets. In the most popular set theories today, and in this book, we do make an assumption about the non-sets, namely, that there are not any at all (among the things we choose to consider). Speaking roughly, here are some reasons for making this assumption:

As we will soon see precisely, there is nevertheless an empty set, $\emptyset$; also the set $\{\emptyset\}$ whose only member is $\emptyset$; the set $\{\emptyset, \{\emptyset\}\}$; the infinite list of sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\{\emptyset\}\}\}$, etc.; the infinite set containing all of these latter, and so on and so on. One (interesting but not decisive) argument for considering no non-sets (and thus as is said only 'pure' sets) goes as follows: In mathematics we are only concerned with form, and thus with, for example, a group only up to isomorphism. Every group obtained starting with non-sets is isomorphic to one in our world of pure sets; so nothing is lost, perhaps, in considering only pure sets.

In fact, there are times when set theory allowing non-sets is useful, and this theory is not completely covered by our theory of pure sets. But, in practice, it is usually very easy to develop any needed part of set theory with non-sets, if one already knows pure set theory. Also, allowing non-sets causes a lot of trivial difficulties which obscure the things which really matter. So as a practical matter, it is convenient to consider, as we do here, only pure sets (henceforth called plain 'sets.') We may still give informal examples involving non-sets.

Henceforth, letters 'x', 'y', 'A', 'B', etc., etc., all range over sets (or really, 'pure sets'). Hence, it is optional, for example, whether one says 'for any A' or 'for any set A.'

We understand as in ordinary English the statement 'x belongs to A' or the synonymous 'x is a member of A', which we abbreviate by: $x \in A$.

Suppose C is the set of all American citizens and D is the set of all American citizens less than twelve feet tall. Then C and D have exactly the same members but have been defined in different ways. Ordinary usage does not take a definitive stand as to whether C and D should be considered to be identical. In mathematical usage, ambiguities are not tolerated. If both usages seems interesting, mathematics just studies both, using two names! In mathematical usage it is generally agreed to use the words 'set' and 'belongs to' in such a way that the sets C and D above *are* considered identical. This convention is expressed systematically in what is called the

Axiom of Extensionality. *If, for any x, $x \in A$ if and only if $x \in B$, then $A = B$.*

(One says '*extensional*' because the different *intensions*, as, say, in our descriptions above of C and D, are being disregarded.) Note that we will mention *some* axioms in the *non-axiomatic* development of Chapters 1-5 (as was always done historically). But in non-axiomatic or intuitive set theory one does not work *only* from the axioms mentioned but allows other intuitions to be used.

We say that A is a *subset* of B, or *A is included in B*, or, in symbols, $A \subseteq B$ if every member of A is a member of B. It follows at once that, in general: $A \subseteq A$ (inclusion is 'reflexive'); if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ (inclusion is 'transitive'). Moreover, the Extensionality Axiom can now be written: if $A \subseteq B$ and $B \subseteq A$ then $A = B$ (inclusion is 'antisymmetric').

Once in a while, instead of discussing sets (which are 'somewhere else'!) we discuss our own language, say, chalkmarks on the blackboard (which are 'in this room'). All expressions in the language of mathematics can be divided (in an extremely important way) into three classes: (1) *asserting* expressions; (2) *naming* expressions; and (3) 'neither of these': The reader will at once be able to classify in this way the following expressions: $x +$; $u + 2$; $x < y$; the unique y such that $x + y = 3$; the set of all x such that $x < 2$; $< x + 3$; $x < y$ or $x < z$; and $\int_1^2 x^2 dx$. Also (using logical symbols) $x < \iota y\,(y^3 = 2)$; $\forall zz$; $\exists! x$ (x is the Queen of England). (The answers are: 3, 2, 1, 2, 2, 3, 1, 2, 1, 3, 1.)

A key notion of set theory is that of forming *the set of all x such that* ..., which is abbreviated $\{x: \ldots\}$. (The *symbols* $\{-: ---\}$ are sometimes called the set-builder. Note to reader: Always read $\{x: \ldots\}$ in full as "the set of all x such that ...".) For example, one can form $\{x: x = 1 \text{ or } x = 2\}$ and $\{x: x \text{ is a real number}\}$, etc.

Clearly, in the expression "$\{x: \ldots\}$", the dots are always to be replaced by an *asserting expression*. (Thus this grammatical notion is just what is needed here!). An extension of this usage is $\{---: \ldots\}$ as in $\{n^3 + n: n \text{ is a positive integer}\}$ which is read: the set of all $n^3 + n$ such that n is a positive integer. In $\{---: \ldots\}$, the '...' is still to be asserting, but the '---' is clearly to be a naming expression. These two set-builder notions do not have to be taken as primitive (or governed by the English meaning), but can be introduced in a simple way *by definition*, as follows:

Definition 1.1.

(a) $\{x: \mathcal{P}x\}$ = *the unique* A *such that for any* x, $x \in A$ *if and only if* $\mathcal{P}x$.
(b) $\{\mathcal{A}_x: \mathcal{P}x\}$ = $\{u$: *for some* x, $u = \mathcal{A}_x$ *and* $\mathcal{P}x\}$.

In Definition 1.1 we are using in a special, familiar way, the capital script letters '$\mathcal{P}$', '$\mathcal{Q}$', etc. and also in a related but different way the capital script letters '$\mathcal{A}$', '$\mathcal{B}$', etc. By properly using these letters, we can avoid the awkward and unclear '. . .' and '− − −' which we used above. Sometimes '$\mathcal{P}$' (for example) has, say, *two* places instead of one as in 1.1, and then one writes '$\mathcal{P}ab$', '$\mathcal{P}xx$', etc. Of course, for example, '$\mathcal{P}x$' is an *asserting expression*; while in 1.1(b) (and always) '$\mathcal{A}_x$' is a *naming expression*.

The letters '$\mathcal{P}$', '$\mathcal{Q}$' are called *predicate* (or sometimes, *class*) *variables*; while '$\mathcal{A}$', '$\mathcal{B}$', etc. are called *operator variables*. Our old letters 'x', 'y', 'x'', $\cdots$, 'A', 'B', $\cdots$ are commonly called *ordinary variables* or, simply, *variables*.

As usual, if we accept as valid a statement involving, say, the predicate variable '$\mathcal{P}$', like 1.1(a) above, then we also accept as valid each *instance* of our statement, that is, each statement obtained from the original one by substituting for '$\mathcal{P}$' a particular asserting expression. (For example, in 1.1(a) one might replace $\mathcal{P}x$ throughout by, say, $\exists u(x \in u \wedge u \in B)$. Similarly (in forming an instance) one can replace, e.g., '$\mathcal{A}$' in a given statement by a particular naming expression (say, replace $\mathcal{A}_x$ by $x^2 + xy$).

We consider now the actual meaning of $\{x: \mathcal{P}x\}$, that is: the unique set A such that for any x, $x \in A$ iff $\mathcal{P}x\}$. Clearly, by Extensionality, *there is always at most one such* A. The famous 'paradoxes' of set theory (cf. §2) involve cases of $\mathcal{P}x$ where there is no such A, i.e., $\{x: \mathcal{P}x\}$ does not exist! In *intuitive* set theory we adopt the (dubious) attitude that we will assume that $\{x: \mathcal{P}x\}$ does exist in all cases that seem natural (and hence all that we shall consider without comment); and we will imagine that our good sense will keep us from trying to form $\{x: \mathcal{P}x\}$ in any 'bad' case!

We will now discuss much more straightforward things, namely, the so-called Boolean operations on sets. (These were first studied by George Boole in 1847 and by A. de Morgan in the same period.) We define:

$$\begin{cases} A \cup B = \{x: x \in A \text{ or } x \in B\}, \text{ called the } \textit{union of } A \text{ and } B. \\ A \cap B = \{x: x \in A \text{ and } x \in B\}, \text{ the } \textit{intersection} \text{ of } A \text{ and } B. \\ A - B = \{x: x \in A \text{ and } x \notin B\}, A \textit{ minus } B. \\ A \ominus B = (A - B) \cup (B - A), \text{ the } \textit{symmetric difference} \text{ of } A \text{ and } B. \\ \quad \emptyset = \{x: x \neq x\}, \text{ the } \textit{empty set} \text{ (clearly by Extensionality} \\ \quad\quad \text{the unique set with no members).} \end{cases}$$

$X - A$ is also written $\tilde{A}^{(X)}$ (and called: the *complement of A with respect to*

X). Often we are in an extended discussion in which all sets A, B, C, etc., being considered are subsets of a particular set X (a 'temporary universe'). For example, X might be the set of real numbers or some other 'space'. Then we write $\tilde{A}^{(X)}$ *simply as* $\tilde{A}$ (and call it simply the *complement of* A). We shall soon see (in §2) that there is no absolute universe (set of all sets). It follows that there is no absolute complement of any set!

Picturing A and B as regions in the plane, the Boolean Operations appear as in the parts of Figure 1 (called *Venn diagrams*). Each shaded area represents the set resulting from the indicated Boolean operation.

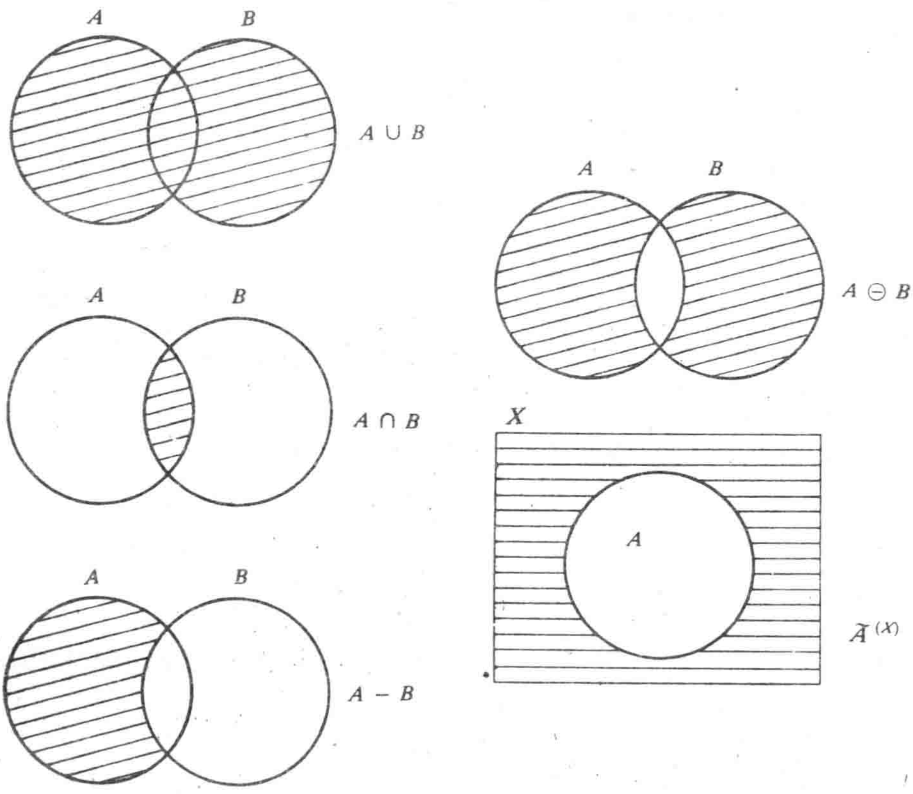

Figure 1

Notice, in the definitions of $A \cup B$, $A \cap B$, etc., above, the close parallel between the Boolean set operations $\sim$, $\cup$, $\cap$ and the logical symbolic $\sim$ (not), $\vee$, $\wedge$. Indeed even the common *symbols* have been chosen so that '$\cup$' resembles '$\vee$', etc.!

We now list some of the laws (all very easily proved) regarding these operations on sets. If a law involves only $\sim$, $\cup$, $\cap$, $\emptyset$, X (not $\ominus$, etc.), its *dual* is obtained simply by replacing these, respectively, by $\sim$, $\cap$, $\cup$, X, $\emptyset$.

Proposition 1.2. *Assume $A, B, C \subseteq X$ and write $\tilde{U}$ for $X - U$.*

(a) $A \cup B = B \cup A$ *and dual.*
$A \cup (B \cup C) = (A \cup B) \cup C$ *and dual.*
$A \cup \emptyset = A$ *and dual* ($A \cap X = A$).
$A \cap \emptyset = \emptyset$ *and dual.*
$\tilde{\tilde{A}} = A$.
$A \subseteq B$ *if and only if* $\tilde{A} \supseteq \tilde{B}$.
$A - B = A \cap \tilde{B}$.
$A \cap B = \emptyset$ *if and only if* $A \subseteq \tilde{B}$.

A and B are called disjoint, *written* $A)(B$, *if* $A \cap B = \emptyset$.

These laws are all obvious. Nevertheless, many mathematicians have 'memorized' at least the last three.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, *and dual.*

Often one compares $\cup$ to $+$ (among numbers) and $\cap$ to $\cdot$. Then the first part of (b) is an analogue of the usual distributive law. But the dual distributive law is quite new (fails for $+$ and $\cdot$) and distinctive algebraically. So the analogy is not too good. Again, most mathematicians have memorized the following laws:

(c) (*De Morgan's law*) $\widetilde{A \cup B} = \tilde{A} \cap \tilde{B}$, *and dual* (*One can also write*: $C - (A \cup B) = (C - A) \cap (C - B)$.)

(d) (*Another distributive law*) $A \cap (B - C) = (A \cap B) - (A \cap C)$
Finally, there is another, better analogy:

(e) *The family of all subsets of X, under $\ominus$ for $+$ and $\cap$ for $\cdot$, with $\emptyset$ for 0 and X for 1, forms a commutative ring with unit. The ring also satisfies the two additional laws: $A + A = 0$ (every element is of order 2) and $A \cdot A = A$ ($\cdot$ is idempotent). (Such a ring is called a Boolean ring with unit.)*

Part (e) is the same as a list of seven or eight simple laws, one being $A \ominus (B \ominus C) = (A \ominus B) \ominus C$.

The laws in Proposition 1.2 can be reinterpreted as laws in *logic* and again as laws in the theory of *switching circuits*. For such simple laws they have great importance. In recent times, for example, computer science has given