CONTENTS

CHAPTER 1

INTRODUCTION

Compressible flows are encountered in many applications in Aerospace and Mechanical engineering. Some examples are flows in nozzles, compressors, turbines and diffusers. In aerospace engineering, in addition to these examples, compressible flows are seen in external aerodynamics, aircraft and rocket engines. In almost all of these applications, air (or some other gas or mixture of gases) is the working fluid. However, steam can be the working substance in turbomachinery applications. Thus, the range of engineering applications in which compressible flow occurs is quite large and hence a clear understanding of the dynamics of compressible flow is essential for engineers.

1.1 Compressibility of Fluids

All fluids are compressible to some extent or other. The compressibility of a fluid is defined as

$$\tau = -\frac{1}{v}\frac{\partial v}{\partial P},\qquad(1.1)$$

where v is the specific volume and P is the pressure. The change in specific

volume corresponding to a given change in pressure, will, of course, depend upon the compression process. That is, for a given change in pressure, the change in specific volume will be different between an isothermal and an adiabatic compression process.

The definition of compressibility actually comes from thermodynamics. Since the specific volume $v = v(T, P)$, we can write

$$dv = \left(\frac{\partial v}{\partial P}\right)_T dP + \left(\frac{\partial v}{\partial T}\right)_P dT \,.$$

From the first term, we can define the isothermal compressibility as $-\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T$ and, from the second term, we can define the coefficient of volume expansion as $\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P$. The second term represents the change in specific volume (or equivalently density) due to a change in temperature. For example, when a gas is heated at constant pressure, the density decreases and the specific volume increases. This change can be large, as is the case in most combustion equipment, without necessarily having any implications on the compressibility of the fluid. It thus follows that compressibility effect is important only when the change in specific volume (or equivalently density) is due largely to a change in pressure.

If the above equation is written in terms of the density ρ, we get

$$\tau = \frac{1}{\rho}\frac{\partial \rho}{\partial P} \,, \tag{1.2}$$

The isothermal compressibility of water and air under standard atmospheric conditions are $5 \times 10^{-10} m^2/N$ and $10^{-5} m^2/N$. Thus, water (in liquid phase) can be treated as an incompressible fluid in all applications. On the contrary, it would seem that, air, with a compressibility that is five orders of magnitude higher, has to be treated as a compressible fluid in all applications. Fortunately, this is not true when flow is involved.

1.2   Compressible and Incompressible Flows

It is well known from high school physics that sound (pressure waves) propagates in any medium with a speed which depends on the bulk compressibility. The less compressible the medium, the higher the speed of sound. Thus, speed of sound is a convenient reference speed, when flow is involved. Speed of sound in air under normal atmospheric conditions is 330 m/s. The implications of this when there is flow are as follows. Let us say that we are considering the flow of air around an automobile travelling at 120 kph (about 33 m/s). This speed is 1/10th of the speed of sound. In other words, compared with 120 kph, sound waves travel 10 times faster. Since the speed of sound *appears* to be high compared with the highest velocity in

the flow field, the medium behaves as though it were incompressible. As the flow velocity becomes comparable to the speed of sound, compressibility effects become more prominent. In reality, the speed of sound itself can vary from one point to another in the flow field and so the velocity at each point has to be compared with the speed of sound at that point. This ratio is called the Mach number, after Ernst Mach who made pioneering contributions in the study of the propagation of sound waves. Thus, the Mach number at a point in the flow can be written as

$$M = \frac{u}{a}, \tag{1.3}$$

where u is the velocity magnitude at any point and a is the speed of sound at that point.

We can come up with a quantitative criterion to give us an idea about the importance of compressibility effects in the flow by using simple scaling arguments as follows. From Bernoulli's equation for steady flow, it follows that $\Delta P \sim \rho U^2$, where U is the characteristic speed. It will be shown in the next chapter that the speed of sound $a = \sqrt{\Delta P / \Delta \rho}$, wherein ΔP and $\Delta \rho$ correspond to an isentropic process. Thus,

$$\frac{\Delta \rho}{\rho} = \frac{1}{\rho} \frac{\Delta \rho}{\Delta P} \Delta P = \frac{U^2}{a^2} = M^2. \tag{1.4}$$

On the other hand, upon rewriting Eqn. 1.2 for an isentropic process, we get

$$\frac{\Delta \rho}{\rho} = \tau_{isentropic} \Delta P.$$

Comparison of these two equations shows clearly that, in the presence of a flow, density changes are proportional to the square of the Mach number[†]. It is customary to assume that the flow is essentially incompressible if the change in density is less than 10% of the mean value[‡]. It thus follows that compressibility effects are significant only when the Mach number exceeds 0.3.

1.3 Perfect Gas Equation of State

In this text, we assume throughout that air behaves as a perfect gas. The equation of state can be written as

$$Pv = RT, \tag{1.5}$$

[†] This is true for steady flows only. For unsteady flows, density changes are proportional to the Mach number.
[‡] Provided the change is predominantly due to a change in pressure.

where T is the temperature[§]. R is the particular gas constant and is equal to $\mathcal{R}/\mathcal{M}$ where $\mathcal{R} = 8314\ J/kmol/K$ is the Universal Gas Constant and $\mathcal{M}$ is the molecular weight in units of $kg/kmol$. Equation 1.5 can be written in many different forms depending upon the application under consideration. A few of these forms are presented here for the sake of completeness. Since the specific volume $v = 1/\rho$, we can write

$$P = \rho RT\ ,$$

or, alternatively, as

$$PV = mRT\ ,$$

where m is the mass and V is the volume. If we define the concentration c as $(m/\mathcal{M})(1/V)$, then,

$$P = c\mathcal{R}T\ . \tag{1.6}$$

Here c has units of $kmol/m^3$. The mass density ρ can be related to the particle density n ($particles/m^3$) through the relationship $\rho = n\mathcal{M}/N_A$. Here we have used the fact that 1 $kmol$ of any substance contains Avogadro number of molecules ($N_A = 6.023 \times 10^{26}$). Thus

$$P = n\frac{\mathcal{R}}{N_A}T\ =\ nk_BT\ , \tag{1.7}$$

where k_B is the Boltzmann constant.

1.3.1 Continuum Hypothesis

In our discussion so far, we have tacitly assumed that properties such as pressure, density, velocity and so on can be evaluated without any ambiguity. While this is intuitively correct, it deserves a closer examination.

Consider the following thought experiment. A cubical vessel of a side dimension L contains a certain amount of a gas. One of the walls of the vessel has a view port to allow observations of the contents within a fixed observation volume. We now propose to measure the density of the gas at an instant as follows - count the number of molecules within the observation volume; multiply this by the mass of each molecule and then divide by the observation volume.

To begin with, let there be 100 molecules inside the vessel. We would notice that the

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[§]In later chapters this will be referred to as the static temperature

density values measured in the aforementioned manner fluctuate wildly going down even to zero at some instants. If we increase the number of molecules progressively to 10^3, 10^4, 10^5 and so on, we would notice that the fluctuations begin to diminish and eventually die out altogether. Increasing the number of molecules beyond this limit would not change the measured value for the density.

We can carry out another experiment in which we attempt to measure the pressure using a pressure sensor mounted on one of the walls. Since the pressure exerted by the gas is the result of the collisions of the molecules on the walls, we would notice the same trend as we did with the density measurement. That is, the pressure measurements too exhibit fluctuations when there are few molecules and the fluctuations die out with increasing number of molecules. The measured value, once again, does not change when the number of molecules is increased beyond a certain limit.

We can intuitively understand that, in both these experiments, when the number of molecules is less, the molecules travel freely for a considerable distance before encountering another molecule or a wall. As the number of molecules is increased, the distance that a molecule on an average can travel between collisions (which is termed as the mean free path, denoted usually by λ) decreases as the collision frequency increases. Once the mean free path decreases below a limiting value, measured property values do not change any more. The gas is then said to behave as a *continuum*. The determination of whether the actual value for the mean free path is small or not has to be made relative to the physical dimensions of the vessel. For instance, if the vessel is itself only about 1 μm in dimension in each side, then a mean free path of 1 μm is not at all small! Accordingly, a parameter known as the Knudsen number (Kn) which is defined as the ratio of the mean free path (λ) to the characteristic dimension (L) is customarily used. Continuum is said to prevail when $Kn \ll 1$. In reality, once the Knudsen number exceeds 10^{-2} or so, the molecules of the gas cease to behave as a continuum.

It is well known from kinetic theory of gases that the mean free path is given as

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n} ,$$
(1.8)

where d is the diameter of the molecule and n is the number density.

EXAMPLE 1.1

Determine whether continuum prevails in the following two practical situations: (a) an aircraft flying at an altitude of 10 km where the ambient pressure and temperature are 26.5 kPa and 230 K respectively and (b) a hypersonic cruise vehicle flying at an altitude of 32 km where the ambient pressure and temperature are 830 Pa and 230 K respectively. Take $d = 3.57 \times 10^{-10}$ m.

Solution In both the cases, it is reasonable to assume the characteristic dimension L to be 1 m.

(a) Upon substituting the given values of the ambient pressure and temperature into the equation of state, $P = nk_BT$, we get $n = 8.34 \times 10^{24} \ particles/m^3$. Hence

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n} = 2.12 \times 10^{-7} \ \text{m} \,.$$

Therefore, the Knudsen number $Kn = \lambda/L = 2.12 \times 10^{-7}$.

(b) Following the same procedure as before, we can easily obtain $Kn = 6.5 \times 10^{-6}$.

It is thus clear that, in both cases, it is quite reasonable to assume that continuum prevails.

1.4 Calorically Perfect Gas

In the study of compressible flows, we need, in addition to the equation of state, an equation relating the internal energy to other measurable properties. The internal energy, strictly speaking, is a function of two thermodynamic properties, namely, temperature and pressure. In reality, the dependence on pressure is very weak for gases and hence is usually neglected. Such gases are called *thermally perfect* and for them $e = f(T)$. The exact nature of this function is examined next.

From a molecular perspective, it can be seen intuitively that the internal energy will depend on the number of modes in which energy can be stored (also known as degrees of freedom) by the molecules (or atoms) and the amount of energy that can be stored in each mode. For monatomic gases, the atoms have the freedom to move (and hence store energy in the form of kinetic energy) in any of the three coordinate directions.

For diatomic gases, assuming that the molecules can be modelled as "dumb bells", additional degrees of freedom are possible. These molecules, in addition to translational motion along the three axes, can also rotate about these axes. Hence, energy storage in the form of rotational kinetic energy is also possible. In reality, since the moment of inertia about the "dumb bell" axis is very small, the amount of kinetic energy that can be stored through rotation about this axis is negligible. Thus, rotation adds essentially two degrees of freedom only. In the "dumb bell" model, the bonds connecting the two atoms are idealized as springs. When the temperature increases beyond 600 K or so, these springs begin to vibrate and so energy can now be stored in the form of vibrational kinetic energy of these springs. When the temperature becomes high ($>$ 2000 K), transition to other electronic levels and dissociation take place and at even higher temperatures the atoms begin to ionize.

These effects do not represent degrees of freedom.

Having identified the number of modes of energy storage, we now turn to the amount of energy that can be stored in each mode. The classical equipartition energy principle states that each degree of freedom, when "fully excited", contributes $1/2\ RT$ to the internal energy per unit mass of the gas. The term "fully excited" means that no more energy can be stored in these modes. For example, the translational mode becomes fully excited at temperatures as low as 3 K itself. For diatomic gases, the rotational mode is fully excited beyond 600 K and the vibrational mode beyond 2000 K or so. Strictly speaking, all the modes are quantized and so the energy stored in each mode has to be calculated using quantum mechanics. However, the spacing between the energy levels for the translational and rotational modes are small enough, that we can assume equipartition principle to hold for these modes.

We can thus write

$$e = \frac{3}{2}RT \ ,$$

for monatomic gases and

$$e = \frac{3}{2}RT + RT + \frac{h\nu/k_BT}{e^{h\nu/k_BT} - 1}RT \ ,$$

for diatomic gases. In the above expression, ν is the fundamental vibrational frequency of the molecule. Note that for large values of T, the last term approaches RT. We have not derived this term formally as it would be well outside the scope of this book. Interested readers may see the book by Anderson for full details.

The enthalpy per unit mass can now be calculated by using the fact that

$$h = e + Pv = e + RT \ .$$

We can calculate C_v and C_p from these equations by using the fact that $C_v = \partial e/\partial T$ and $C_p = \partial h/\partial T$. Thus

$$C_v = \frac{3}{2}R \ ,$$

for monatomic gases and

$$C_v = \frac{5}{2}R + \frac{(h\nu/k_BT)^2 \ e^{h\nu/k_BT}}{\left(e^{h\nu/k_BT} - 1\right)^2}R \ ,$$

for diatomic gases. The variation of C_v/R is illustrated schematically in Fig. 1.1. It is clear from this figure that $C_v = 5/2R$ in the temperature range $50K \leq T \leq 600K$. In this range, $C_p = 7/2R$, and thus the ratio of specific heats $\gamma = 7/5$ for diatomic gases. For monatomic gases, it is easy to show that $\gamma = 5/3$. In