

FOURIER ANALYSIS  
ON FINITE GROUPS  
WITH APPLICATIONS IN  
SIGNAL PROCESSING  
AND SYSTEM DESIGN

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# Fourier Analysis on Finite Groups with Applications in Signal Processing and System Design

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# *Preface*

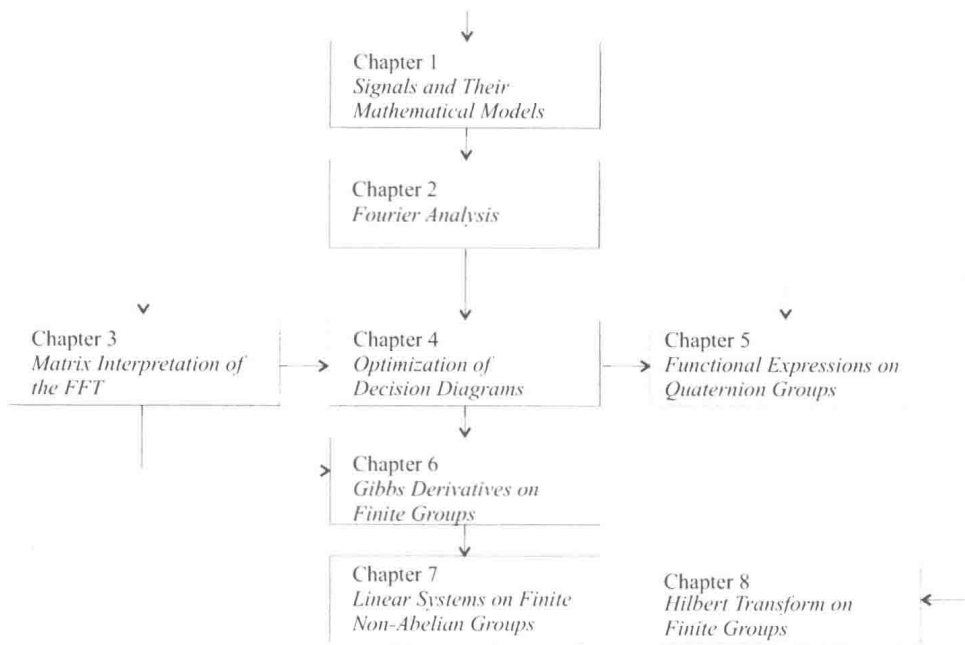
We believe that the group-theoretic approach to spectral techniques and, in particular, Fourier analysis, has many advantages, for instance, the possibility for a unified treatment of various seemingly unrelated classes of signals. This approach allows to extend the powerful methods of classical Fourier analysis to signals that are defined on very different algebraic structures that reflect the properties of the modelled phenomenon.

Spectral methods that are based on finite Abelian groups play a very important role in many applications in signal processing and logic design. In recent years the interest in developing methods that are based on Finite non-Abelian groups has been steadily growing, and already, there are many examples of cases where the spectral methods based only on Abelian groups do not provide the best performance.

This monograph reviews research by the authors in the area of abstract harmonic analysis on finite non-Abelian groups. Many of the results discussed have already appeared in somewhat different forms in journals and conference proceedings.

We have aimed for presenting the results here in a consistent and self-contained way, with a uniform notation and avoiding repetition of well-known results from abstract harmonic analysis, except when needed for derivation, discussion and appreciation of the results. However, the results are accompanied, where necessary or appropriate, with a short discussion including comments concerning their relationship to the existing results in the area.

The purpose of this monograph is to provide a basis for further study in abstract harmonic analysis on finite Abelian and non-Abelian groups and its applications.



**Fig. 0.1** Relationships among the chapters.

The monograph will hopefully stimulate new research that results in new methods and techniques to process signals modelled by functions on finite non-Abelian groups.

Fig. 0.1 shows relationships among the chapters.

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Niš, Dortmund, Tampere

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Prof. Mark G. Karpovsky and Prof. Lazar A. Trachtenberg have traced in a series of publications chief directions in research in Fourier analysis on finite non-Abelian groups. We are following these directions in our research in the area, in particular in extending the theory of Gibbs differentiation to non-Abelian structures. For that, we are very indebted to them both.

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R.S.S., C.M, J.T.A.





# Acronyms

ACDD	Arithmetic transform decision diagram
BDD	Binary decision diagram
BDT	Binary decision tree
DD	Decision diagram
DT	Decision tree
DFT	Discrete Fourier transform
FFT	Fast Fourier transform
FDD	Functional decision diagram
FNADD	Fourier decision diagram on finite non-Abelian groups
FNADT	Fourier decision tree on finite non-Abelian groups
FNAPDD	Fourier decision diagram on finite non-Abelian groups with preprocessing
FNAPDT	Fourier decision tree on finite non-Abelian groups with preprocessing
KDD	Kronecker decision diagram
mvMTDD	Matrix-valued multi-terminal decision diagram
MTBDD	Multi-terminal binary decision diagram
MTBDT	Multi-terminal binary decision tree
MDD	Multiple-place diagram
MTDD	Multi-terminal decision diagram
MTDT	Multi-terminal decision tree
nvMTDD	Number-valued multi-terminal decision diagram
PKDD	Pseudo-Kronecker decision diagram
QDD	Quaternary decision diagrams
SBDD	Shared binary decision diagrams
TVFG	Two-variable function generator
WDD	Walsh decision diagram



# Contents

<i>Preface</i>	v
<i>Acknowledgments</i>	vii
<i>Acronyms</i>	xxiii
<b>1 Signals and Their Mathematical Models</b>	<b>1</b>
1.1 Systems	1
1.2 Signals	2
1.3 Mathematical Models of Signals	3
References	6
<b>2 Fourier Analysis</b>	<b>11</b>
2.1 Representations of Groups	12
2.1.1 Complete reducibility	13
2.2 Fourier Transform on Finite Groups	18
2.3 Properties of the Fourier transform	23
2.4 Matrix interpretation of the Fourier transform on finite non-Abelian groups	26
2.5 Fast Fourier transform on finite non-Abelian groups	28
References	35
	<b>ix</b>

<b>3</b>	<b>Matrix Interpretation of the FFT</b>	<b>37</b>
3.1	<i>Matrix interpretation of FFT on finite non-Abelian groups</i>	38
3.2	<i>Illustrative examples</i>	41
3.3	<i>Complexity of the FFT</i>	59
3.3.1	<i>Complexity of calculations of the FFT</i>	62
3.3.2	<i>Remarks on programming implementation of FFT</i>	66
3.4	<i>FFT through decision diagrams</i>	66
3.4.1	<i>Decision diagrams</i>	66
3.4.2	<i>FFT on finite non-Abelian groups through DDs</i>	68
3.4.3	<i>MTDDs for the Fourier spectrum</i>	76
3.4.4	<i>Complexity of DDs calculation methods</i>	76
	<i>References</i>	80
<b>4</b>	<b>Optimization of Decision Diagrams</b>	<b>85</b>
4.1	<i>Reduction Possibilities in Decision Diagrams</i>	86
4.2	<i>Group-theoretic Interpretation of DD</i>	93
4.3	<i>Fourier Decision Diagrams</i>	96
4.3.1	<i>Fourier decision trees</i>	96
4.3.2	<i>Fourier decision diagrams</i>	107
4.4	<i>Discussion of Different Decompositions</i>	108
4.4.1	<i>Algorithm for optimization of DDs</i>	110
4.5	<i>Representation of Two-Variable Function Generator</i>	110
4.6	<i>Representation of adders by Fourier DD</i>	114
4.7	<i>Representation of multipliers by Fourier DD</i>	117
4.8	<i>Complexity of FNADD</i>	123
4.9	<i>Fourier DDs with Preprocessing</i>	129
4.9.1	<i>Matrix-valued functions</i>	129
4.9.2	<i>Fourier transform for matrix-valued functions</i>	130
4.10	<i>Fourier Decision Trees with Preprocessing</i>	135
4.11	<i>Fourier Decision Diagrams with Preprocessing</i>	136
4.12	<i>Construction of FNAPDD</i>	137
4.13	<i>Algorithm for Construction of FNAPDD</i>	151
4.13.1	<i>Algorithm for representation</i>	152
4.14	<i>Optimization of FNAPDD</i>	153
	<i>References</i>	154

<b>5</b>	<b>Functional Expressions on Quaternion Groups</b>	<b>157</b>
5.1	<i>Fourier expressions on finite dyadic groups</i>	158
5.1.1	<i>Finite dyadic groups</i>	158
5.2	<i>Fourier Expressions on <math>Q_2</math></i>	158
5.3	<i>Arithmetic Expressions</i>	160
5.4	<i>Arithmetic expressions from Walsh expansions</i>	161
5.5	<i>Arithmetic expressions on <math>Q_2</math></i>	163
5.5.1	<i>Arithmetic expressions and arithmetic-Haar expressions</i>	166
5.5.2	<i>Arithmetic-Haar expressions and Kronecker expressions</i>	166
5.6	<i>Different Polarity Polynomial Expressions</i>	167
5.6.1	<i>Fixed-polarity Fourier expansions in <math>C(Q_2)</math></i>	168
5.6.2	<i>Fixed-polarity arithmetic-Haar expressions</i>	169
5.7	<i>Calculation of the arithmetic-Haar coefficients</i>	172
5.7.1	<i>FFT-like algorithm</i>	172
5.7.2	<i>Calculation of arithmetic-Haar coefficients through decision diagrams</i>	174
	<i>References</i>	180
<b>6</b>	<b>Gibbs Derivatives on Finite Groups</b>	<b>183</b>
6.1	<i>Definition and properties of Gibbs derivatives on finite non-Abelian groups</i>	184
6.2	<i>Gibbs anti-derivative</i>	186
6.3	<i>Partial Gibbs derivatives</i>	187
6.4	<i>Gibbs differential equations</i>	189
6.5	<i>Matrix interpretation of Gibbs derivatives</i>	190
6.6	<i>Fast algorithms for calculation of Gibbs derivatives on finite groups</i>	192
6.6.1	<i>Complexity of Calculation of Gibbs Derivatives</i>	198
6.7	<i>Calculation of Gibbs derivatives through DDs</i>	201
6.7.1	<i>Calculation of partial Gibbs derivatives</i>	203
	<i>References</i>	207
<b>7</b>	<b>Linear Systems on Finite Non-Abelian Groups</b>	<b>211</b>
7.1	<i>Linear shift-invariant systems on groups</i>	211
7.2	<i>Linear shift-invariant systems on finite non-Abelian groups</i>	213

7.3	<i>Gibbs derivatives and linear systems</i>	214
7.3.1	<i>Discussion</i>	215
	<i>References</i>	217
8	<b>Hilbert Transform on Finite Groups</b>	221
8.1	<i>Some results of Fourier analysis on finite non-Abelian groups</i>	223
8.2	<i>Hilbert transform on finite non-Abelian groups</i>	227
8.3	<i>Hilbert transform in finite fields</i>	231
	<i>References</i>	234
	<i>Index</i>	235

# *List of Figures*

0.1	Relationships among the chapters.	vi
2.1	FFT on the quaternion group $Q_2$ .	32
2.2	Flow-graph for FFT algorithm for the inverse Fourier transform on $Q_8$ .	33
2.3	FFT on the dyadic group of order 8.	34
3.1	Structure of the flow-graph of the FFT on the group $G_{2 \times 8}$ .	46
3.2	Structure of the flow-graph for FFT on the group $G_{32}$ .	48
3.3	Structure of the flow-graph for FFT on the group $G_{32}$ through a part of fast Walsh transform.	49
3.4	Structure of the flow-graph for FFT on the group $G_{32}$ using FFT on $Q_2$ .	50
3.5	Structure of the flow-graph for FFT on the group $G_{6 \times 6}$ .	53
3.6	Structure of the flow-graph for FFT on the group $G_{3 \times 6}$ .	57
3.7	Structure of the flow-graph for FFT on $G_{24}$ .	58
3.8	Structure of the flow-graph for FFT on $S_3$ .	60
3.9	Structure of the flow-graph for FFT on $G_{24}$ with FFT on $S_3$ .	61



3.10	Number of operations in FFT.	63
3.11	Time requirements.	65
3.12	Memory requirements.	65
3.13	MTDD for $f$ in Example 3.6.	68
3.14	BDD for $f$ in Example 3.7.	69
3.15	MTBDD for the Walsh spectrum for $f$ in Example 3.7.	72
3.16	Calculation procedure for the Fourier transform.	73
3.17	Calculation of the Fourier spectrum through MTDD.	77
3.18	nvMTDD for the Fourier spectrum for $f$ in Example 3.9.	78
4.1	Shannon tree for $n = 4$ .	87
4.2	Subtrees in the Shannon tree for pairs of variables $(x_1, x_2), (x_3, x_4)$ .	88
4.3	(a) Subtree in the Shannon tree for a pair of variables $(x_i, x_{i+1})$ , (b) QDD non-terminal node.	89
4.4	QDD for $n = 4$ .	89
4.5	Subtrees in the Shannon tree for $x_1, (x_2, x_3, x_4)$ .	90
4.6	(a) Subtree in the Shannon tree for $(x_{i-1}, x_i, x_{i+1})$ , (b) Non-terminal node with eight outgoing edges.	90
4.7	Decision tree with nodes with two and eight outgoing edges.	91
4.8	Subtree with the Shannon $S_2$ and QDD non-terminal nodes.	91
4.9	Decision tree for $n = 4$ with nodes with two and four outgoing edges.	92
4.10	Decomposition of the domain group $G_{16} = C_2^4$ .	93
4.11	Decomposition of the domain group $G_{16} = C_4^2$ .	94
4.12	Fourier decision diagram for $f$ in Example 4.1.	100
4.13	Complex-valued FNADD for $Q_2$ .	100
4.14	Decision tree with the Shannon node $S_2$ and FNADD nodes for $Q_2$ .	101
4.15	One-level multi-terminal decision tree of $f$ in Example 4.3 on $G_6$ .	103