

Undergraduate Texts in Mathematics

Peter Petersen

Linear Algebra

线性代数

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Peter Petersen



Linear Algebra

This book covers the aspects of linear algebra and its applications in both theoretical and applied contexts. It is designed for students in mathematics, physics, and engineering. The book includes chapters on vector spaces, linear transformations, eigenvalues, and the Jordan normal form. It also covers the spectral theorem and the minimal polynomial. The book is written in a clear and concise style, with many examples and exercises. It is a valuable resource for anyone interested in linear algebra.

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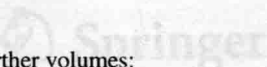
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Linear Algebra

by Peter Petersen

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Preface

This book covers the aspects of linear algebra that are included in most advanced undergraduate texts. All the usual topics from complex vector spaces, complex inner products the spectral theorem for normal operators, dual spaces, quotient spaces, the minimal polynomial, the Jordan canonical form, and the Frobenius (or rational) canonical form are explained. A chapter on determinants has been included as the last chapter, but they are not used in the text as a whole. A different approach to linear algebra that does not use determinants can be found in [Axler].

The expected prerequisites for this book would be a lower division course in matrix algebra. A good reference for this material is [Bretscher].

In the context of other books on linear algebra it is my feeling that this text is about on a par in difficulty with books such as [Axler, Curtis, Halmos, Hoffman-Kunze, Lang]. If you want to consider more challenging texts, I would suggest looking at the graduate level books [Greub, Roman, Serre].

Chapter 1 contains all of the basic material on abstract vector spaces and linear maps. The dimension formula for linear maps is the theoretical highlight. To facilitate some more concrete developments we cover matrix representations, change of basis, and Gauss elimination. Linear independence which is usually introduced much earlier in linear algebra only comes towards to the end of the chapter. But it is covered in great detail there. We have also included two sections on dual spaces and quotient spaces that can be skipped.

Chapter 2 is concerned with the theory of linear operators. Linear differential equations are used to motivate the introduction of eigenvalues and eigenvectors, but this motivation can be skipped. We then explain how Gauss elimination can be used to compute the eigenvalues as well as the eigenvectors of a matrix. This is used to understand the basics of how and when a linear operator on a finite-dimensional space is diagonalizable. We also introduce the minimal polynomial and use it to give the classic characterization of diagonalizable operators. In the later sections we give a fairly simple proof of the Cayley–Hamilton theorem and the cyclic subspace decomposition. This quickly leads to the Frobenius canonical form. This canonical form is our most general result on how to find a simple matrix representation for a linear map in case it is not diagonalizable. The antepenultimate section explains

how the Frobenius canonical form implies the Jordan–Chevalley decomposition and the Jordan–Weierstrass canonical form. In the last section, we present a quick and elementary approach to the Smith normal form. This form allows us to calculate directly all of the similarity invariants of a matrix using basic row and column operations on matrices with polynomial entries.

Chapter 3 includes material on inner product spaces. The Cauchy–Schwarz inequality and its generalization to Bessel’s inequality and how they tie in with orthogonal projections form the theoretical centerpiece of this chapter. Along the way, we cover standard facts about orthonormal bases and their existence through the Gram–Schmidt procedure as well as orthogonal complements and orthogonal projections. The chapter also contains the basic elements of adjoints of linear maps and some of its uses to orthogonal projections as this ties in nicely with orthonormal bases. We end the chapter with a treatment of matrix exponentials and systems of differential equations.

Chapter 4 covers quite a bit of ground on the theory of linear maps between inner product spaces. The most important result is of course the spectral theorem for self-adjoint operators. This theorem is used to establish the canonical forms for real and complex normal operators, which then gives the canonical form for unitary, orthogonal, and skew-adjoint operators. It should be pointed out that the proof of the spectral theorem does not depend on whether we use real or complex scalars nor does it rely on the characteristic or minimal polynomials. The reason for ignoring our earlier material on diagonalizability is that it is desirable to have a theory that more easily generalizes to infinite dimensions. The usual proofs that use the characteristic and minimal polynomials are relegated to the exercises. The last sections of the chapter cover the singular value decomposition, the polar decomposition, triangulability of complex linear operators (Schur’s theorem), and quadratic forms.

Chapter 5 covers determinants. At this point, it might seem almost useless to introduce the determinant as we have covered the theory without needing it much. While not indispensable, the determinant is rather useful in giving a clean definition for the characteristic polynomial. It is also one of the most important invariants of a finite-dimensional operator. It has several nice properties and gives an excellent criterion for when an operator is invertible. It also comes in handy in giving a formula (Cramer’s rule) for solutions to linear systems. Finally, we discuss its uses in the theory of linear differential equations, in particular in connection with the variation of parameters formula for the solution to inhomogeneous equations. We have taken the liberty of defining the determinant of a linear operator through the use of volume forms. Aside from showing that volume forms exist, this gives a rather nice way of proving all the properties of determinants without using permutations. It also has the added benefit of automatically giving the permutation formula for the determinant and hence showing that the sign of a permutation is well defined.

An * after a section heading means that the section is not necessary for the understanding of other sections without an *.

Let me offer a few suggestions for how to teach a course using this book. My assumption is that most courses are based on 150 min of instruction per week with

a problem session or two added. I realize that some courses meet three times while others only two, so I will not suggest how much can be covered in a lecture.

First, let us suppose that you, like me, teach in the pedagogically impoverished quarter system: It should be possible to teach Chap. 1, Sects. 1.2–1.13 in 5 weeks, being a bit careful about what exactly is covered in Sects. 1.12 and 1.13. Then, spend 2 weeks on Chap. 2, Sects. 2.3–2.5, possibly omitting Sect. 2.4 covering the minimal polynomial if timing looks tight. Next spend 2 weeks on Chap. 3, Sects. 3.1–3.5, and finish the course by covering Chap. 4, Sect. 4.1 as well as Exercise 9 in Sect. 4.1. This finishes the course with a proof of the Spectral Theorem for self-adjoint operators, although not the proof I would recommend for a more serious treatment.

Next, let us suppose that you teach in a short semester system, as the ones at various private colleges and universities. You could then add 2 weeks of material by either covering the canonical forms from Chap. 2, Sects. 2.6–2.8 or alternately spend 2 weeks covering some of the theory of linear operators on inner product spaces from Chap. 4, Sects. 4.1–4.5. In case you have 15 weeks at your disposal, it might be possible to cover both of these topics rather than choosing between them.

Finally, should you have two quarters, like we sometimes do here at UCLA, then you can in all likelihood cover virtually the entire text. I would certainly recommend that you cover all of Chap. 4 and the canonical form sections in Chap. 2, Sects. 2.6–2.8, as well as the chapter on determinants. If time permits, it might even be possible to include Sects. 2.2, 3.7, 4.8, and 5.8 that cover differential equations.

This book has been used to teach a bridge course on linear algebra at UCLA as well as a regular quarter length course. The bridge course was funded by a VIGRE NSF grant, and its purpose was to ensure that incoming graduate students had really learned all of the linear algebra that we expect them to know when starting graduate school. The author would like to thank several UCLA students for suggesting various improvements to the text: Jeremy Brandman, Sam Chamberlain, Timothy Eller, Clark Grubb, Vanessa Idiarte, Yanina Landa, Bryant Mathews, Shervin Mosadeghi, and Danielle O'Donnol. I am also pleased to acknowledge NSF support from grants DMS 0204177 and 1006677.

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Chapter 1

Basic Theory

In the first chapter, we are going to cover the definitions of vector spaces, linear maps, and subspaces. In addition, we are introducing several important concepts such as basis, dimension, direct sum, matrix representations of linear maps, and kernel and image for linear maps. We shall prove the dimension theorem for linear maps that relate the dimension of the domain to the dimensions of kernel and image. We give an account of Gauss elimination and how it ties in with the more abstract theory. This will be used to define and compute the characteristic polynomial in Chap. 2.

It is important to note that Sects. 1.13 and 1.12 contain alternate proofs of some of the important results in this chapter. As such, some people might want to go right to these sections after the discussion on isomorphism in Sect. 1.8 and then go back to the missed sections.

As induction is going to play a big role in many of the proofs, we have chosen to say a few things about that topic in the first section.

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
2. Every integer ≥ 2 has a prime factorization.
3. Every polynomial has a root.

The first statement is pretty straightforward to understand. The second is a bit more complicated, and we also note that in fact, there is only a statement for each integer ≥ 2 . This could be finessed by saying that each integer $n + 1$, $n \geq 1$ has a prime factorization. This, however, seems too pedantic and also introduces extra and irrelevant baggage by using addition. The third statement is obviously quite different from the other two. For one thing, it only stands a chance of being true if we also assume that the polynomials have degree ≥ 1 . This gives us the idea of how this can be tied to the positive integers. The statement can be paraphrased as: Every polynomial of degree ≥ 1 has a root. Even then, we need to be more precise as not all polynomials have any real roots.

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1.1 Induction and Well-Ordering*

A fundamental property of the *natural numbers*, i.e., the positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$, that will be used throughout the book is the fact that they are *well ordered*. This means that any nonempty subset $S \subset \mathbb{N}$ has a smallest element $s_{\min} \in S$ such that $s_{\min} \leq s$ for all $s \in S$. Using the natural ordering of the integers, rational numbers, or real numbers, we see that this property does not hold for those numbers. For example, the half-open interval $(0, \infty)$ does not have a smallest element.

In order to justify that the positive integers are well ordered, let $S \subset \mathbb{N}$ be nonempty and select $k \in S$. Starting with 1, we can check whether it belongs to S . If it does, then $s_{\min} = 1$. Otherwise, check whether 2 belongs to S . If $2 \in S$ and $1 \notin S$, then we have $s_{\min} = 2$. Otherwise, we proceed to check whether 3 belongs to S . Continuing in this manner, we must eventually find $k_0 \leq k$, such that $k_0 \in S$, but $1, 2, 3, \dots, k_0 - 1 \notin S$. This is the desired minimum: $s_{\min} = k_0$.

We shall use the well-ordering of the natural numbers in several places in this text. A very interesting application is to the proof of the prime factorization theorem: any integer ≥ 2 is a product of prime numbers. The proof works the following way. Let $S \subset \mathbb{N}$ be the set of numbers which do not admit a prime factorization. If S is empty, we are finished; otherwise, S contains a smallest element $n = s_{\min} \in S$. If n has no divisors, then it is a prime number and hence has a prime factorization. Thus, n must have a divisor $p > 1$. Now write $n = p \cdot q$. Since $p, q < n$ both numbers must have a prime factorization. But then also $n = p \cdot q$ has a prime factorization. This contradicts that S is nonempty.

The second important idea that is tied to the natural numbers is that of *induction*. Sometimes, it is also called *mathematical induction* so as not to confuse it with the inductive method from science. The types of results that one can attempt to prove with induction always have a statement that needs to be verified for each number $n \in \mathbb{N}$. Some good examples are

1. $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.
2. Every integer ≥ 2 has a prime factorization.
3. Every polynomial has a root.

The first statement is pretty straightforward to understand. The second is a bit more complicated, and we also note that in fact, there is only a statement for each integer ≥ 2 . This could be finessed by saying that each integer $n + 1$, $n \geq 1$ has a prime factorization. This, however, seems too pedantic and also introduces extra and irrelevant baggage by using addition. The third statement is obviously quite different from the other two. For one thing, it only stands a chance of being true if we also assume that the polynomials have degree ≥ 1 . This gives us the idea of how this can be tied to the positive integers. The statement can be paraphrased as: Every polynomial of degree ≥ 1 has a root. Even then, we need to be more precise as $x^2 + 1$ does not have any real roots.

In order to explain how induction works abstractly, suppose that we have a statement $P(n)$ for each $n \in \mathbb{N}$. Each of the above statements can be used as an example of what $P(n)$ can be. The induction process now works by first ensuring that the anchor statement is valid. In other words, we first check that $P(1)$ is true. We then have to establish the *induction step*. This means that we need to show that if $P(n-1)$ is true, then $P(n)$ is also true. The assumption that $P(n-1)$ is true is called the *induction hypothesis*. If we can establish the validity of these two facts, then $P(n)$ must be true for all n . This follows from the well-ordering of the natural numbers. Namely, let $S = \{n : P(n) \text{ is false}\}$. If S is empty, we are finished, otherwise, S has a smallest element $k \in S$. Since $1 \notin S$, we know that $k > 1$. But this means that we know that $P(k-1)$ is true. The induction step then implies that $P(k)$ is true as well. This contradicts that S is nonempty.

Let us see if we can use this procedure on the above statements. For 1, we begin by checking that $1 = \frac{1(1+1)}{2}$. This is indeed true. Next, we assume that

$$1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)n}{2},$$

and we wish to show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Using the induction hypothesis, we see that

$$\begin{aligned} (1 + 2 + 3 + \cdots + (n-1)) + n &= \frac{(n-1)n}{2} + n \\ &= \frac{(n-1)n + 2n}{2} \\ &= \frac{(n+1)n}{2}. \end{aligned}$$

Thus, we have shown that $P(n)$ is true provided $P(n-1)$ is true.

For 2, we note that two is a prime number and hence has a prime factorization. Next, we have to prove that n has a prime factorization if $(n-1)$ does. This, however, does not look like a very promising thing to show. In fact, we need a stronger form of induction to get this to work.

The induction step in the stronger version of induction is as follows: If $P(k)$ is true for all $k < n$, then $P(n)$ is also true. Thus, the induction hypothesis is much stronger as we assume that all statements prior to $P(n)$ are true. The proof that this form of induction works is virtually identical to the above justification.

Let us see how this stronger version can be used to establish the induction step for 2. Let $n \in \mathbb{N}$, and assume that all integers below n have a prime factorization. If n has no divisors other than 1 and n , it must be a prime number and we are

finished. Otherwise, $n = p \cdot q$ where $p, q < n$. Whence, both p and q have prime factorizations by our induction hypothesis. This shows that also n has a prime factorization.

We already know that there is trouble with statement 3. Nevertheless, it is interesting to see how an induction proof might break down. First, we note that all polynomials of degree 1 look like $ax + b$ and hence have $-\frac{b}{a}$ as a root. This anchors the induction. To show that all polynomials of degree n have a root, we need to first decide which of the two induction hypotheses are needed. There really is not anything wrong by simply assuming that all polynomials of degree $< n$ have a root. In this way, we see that at least any polynomial of degree n that is the product of two polynomials of degree $< n$ must have a root. This leaves us with the so-called prime or irreducible polynomials of degree n , namely, those polynomials that are not divisible by polynomials of degree ≥ 1 and $< n$. Unfortunately, there is not much we can say about these polynomials. So induction does not seem to work well in this case. All is not lost however. A careful inspection of the "proof" of 3 can be modified to show that any polynomial has a prime factorization. This is studied further in Sect. 2.1.

The type of statement and induction argument that we will encounter most often in this text is definitely of the third type. That is to say, it certainly will never be of the very basic type seen in statement 1. Nor will it be as easy as in statement 2. In our cases, it will be necessary to first find the integer that is used for the induction, and even then, there will be a whole collection of statements associated with that integer. This is what is happening in the third statement. There, we first need to select the degree as our induction integer. Next, there are still infinitely many polynomials to consider when the degree is fixed. Finally, whether or not induction will work or is the "best" way of approaching the problem might actually be questionable.

The following statement is fairly typical of what we shall see: Every subspace of \mathbb{R}^n admits a basis with $\leq n$ elements. The induction integer is the dimension n , and for each such integer, there are infinitely many subspaces to be checked. In this case, an induction proof will work, but it is also possible to prove the result without using induction.

Fig. 1.1 Vector addition



Fig. 1.2 Scalar multiplication

1.2 Elementary Linear Algebra

Our first picture of what vectors are and what we can do with them comes from viewing them as geometric objects in the plane and space. Simply put, a vector is an arrow of some given length drawn in the plane. Such an arrow is also known as an oriented line segment. We agree that vectors that have the same length and orientation are equivalent no matter where they are based. Therefore, if we base them at the origin, then vectors are determined by their endpoints. Using a parallelogram, we can add such vectors (see Fig. 1.1). We can also multiply them by scalars. If the scalar is negative, we are changing the orientation. The size of the scalar determines how much we are scaling the vector, i.e., how much we are changing its length (see Fig. 1.2).

This geometric picture can also be taken to higher dimensions. The idea of scaling a vector does not change if it lies in space, nor does the idea of how to add vectors, as two vectors must lie either on a line or more generically in a plane. The problem comes when we wish to investigate these algebraic properties further. As an example, think about the associative law

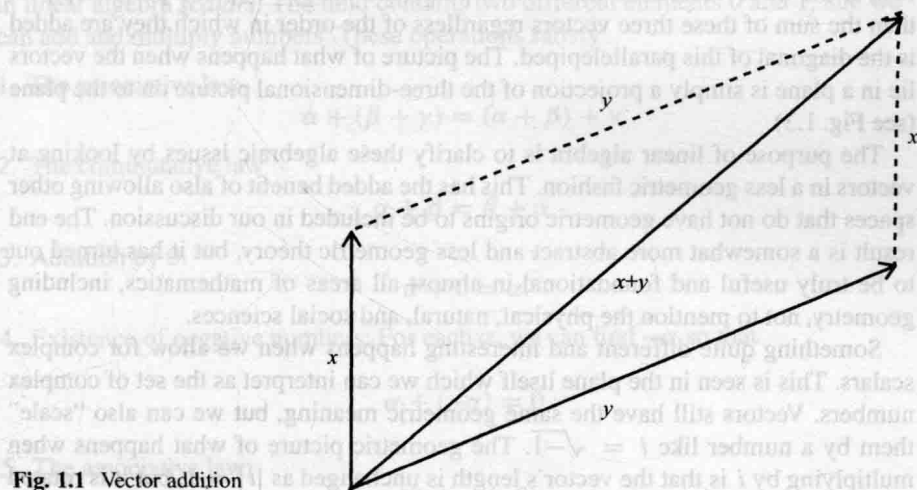


Fig. 1.1 Vector addition

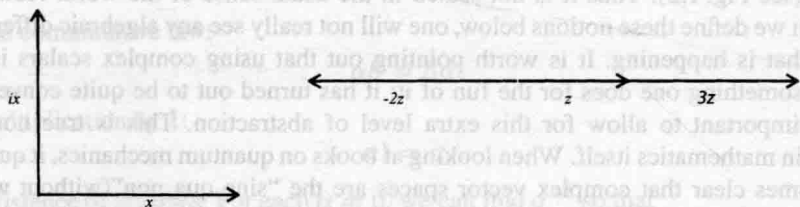


Fig. 1.2 Scalar multiplication