Problem Books in Mathematics

Bernard Gelbaum

Problems in Analysis

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With 9 Illustrations

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Preface

These problems and solutions are offered to students of mathematics who have learned real analysis, measure theory, elementary topology and some theory of topological vector spaces. The current widely used texts in these subjects provide the background for the understanding of the problems and the finding of their solutions. In the bibliography the reader will find listed a number of books from which the necessary working vocabulary and techniques can be acquired.

Thus it is assumed that terms such as *topological space*, *σ-ring*, *metric*, *measurable*, *homeomorphism*, etc., and groups of symbols such as $A \cap B$, $x \in X$, $f: \mathbb{R} \ni x \mapsto x^2 - 1$, etc., are familiar to the reader. They are used without introductory definition or explanation. Nevertheless, the index provides definitions of some terms and symbols that might prove puzzling.

Most terms and symbols peculiar to the book are explained in the various introductory paragraphs titled Conventions. Occasionally definitions and symbols are introduced and explained within statements of problems or solutions.

Although some solutions are complete, others are designed to be sketchy and thereby to give their readers an opportunity to exercise their skill and imagination.

Numbers written in boldface inside square brackets refer to the bibliography.

I should like to thank Professor P. R. Halmos for the opportunity to discuss with him a variety of technical, stylistic, and mathematical questions that arose in the writing of this book.

Buffalo, NY
August 1982

B.R.G.

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Problems
1. Set Algebra

Conventions

The set of positive integers is $\mathbb{N}$; the set of real numbers is $\mathbb{R}$. The set of all subsets of a set X is 2^X. If $\mathsf{E} \subset 2^X$, then $\mathsf{R}(\mathsf{E})$, $(\sigma\mathsf{R}(\mathsf{E})$, $\mathsf{A}(\mathsf{E})$, $\sigma\mathsf{A}(\mathsf{E}))$ is the intersection of the (nonempty) set of rings (σ-rings, algebras, σ-algebras) containing E and contained in 2^X. It is the ring (σ-ring, algebra, σ-algebra) generated by E. The set of x in X such that $\cdots$ is $\{x : \cdots\}$. If $A \subset X$ then $A' = \{x : x \notin A\}$ and if $B \subset X$ then $A \backslash B = A \cap B'$. The cardinality of X is $\text{card}(X)$. If X is a topological space then $\mathsf{O}(X)$ $(\mathsf{F}(X)$, $\mathsf{K}(X))$ is the set of open (closed, compact) subsets of X. A subset M of 2^X is monotone if it is closed with respect to the formation of countable unions and intersections of monotone sequences in M, i.e., if $\{A_n : n = 1, 2, \ldots\}$ is a sequence in M and $A_n \subset A_{n+1}$ $(A_n \supset A_{n+1})$ for n in $\mathbb{N}$ then $\cup_n A_n (\cap_n A_n)$ is in M. The set $\mathsf{M}(\mathsf{E})$ is the monotone subset of 2^X generated by E.

1. Show that if M is monotone and closed with respect to the formation of finite unions and intersections, it is closed with respect to the formation of countable unions and intersections.

2. Show that if M is monotone, R is a ring and $\mathsf{M} \supset \mathsf{R}$, then $\mathsf{M} \supset \sigma\mathsf{R}(\mathsf{R})$.

3. Show that if $2^{\mathbb{R}} \supset \mathsf{M} \supset \mathsf{O}(\mathbb{R})$ and M is monotone then $\mathsf{M} \supset \mathsf{F}(\mathbb{R})$. Repeat, with $\mathbb{R}$ in the preceding sentence replaced by X, a metric space.

4. Show that if $2^{\mathbb{R}} \supset \mathsf{M} \supset \mathsf{O}(\mathbb{R})$ and M is monotone then $\mathsf{M} \supset \sigma\mathsf{R}(\mathsf{O}(\mathbb{R}))$ and $\sigma\mathsf{R}(\mathsf{O}(\mathbb{R})) = \sigma\mathsf{R}(\mathsf{F}(\mathbb{R})) = \sigma\mathsf{R}(\mathsf{K}(\mathbb{R}))$.

5. Show that if S is a σ-ring then $\text{card}(\mathsf{S}) \neq \text{card}(\mathbb{N})$.

6. Show that if $A \in \sigma\mathsf{R}(\mathsf{E})$ then there is in E a finite or countable subset E_0 such that $A \in \sigma\mathsf{R}(\mathsf{E}_0)$.

7. Assume that for each sequence $\{p, q, r, \ldots\}$ of positive integers there is a sequence $\{A_p, B_{pq}, A_{pqr}, \ldots\}$ contained in 2^X. Let the following conditions obtain: i) $A_p = \bigcap_q B_{pq}$, $B_{pq} = \bigcup_r A_{pqr}, \ldots$; ii) for each sequence S of sets $A_p, B_{pq}, A_{pqr}, \ldots$, there is in $\mathbb{N}$ an $m(S)$ such that each member of S with more than $m(S)$ indices is in E. Let A be the set of all countable unions of sets A_p. Show that A is closed with respect to the formation of countable unions and intersections of its members. Assume additionally: iii) if $E \in \mathsf{E}$ then $E' \in \mathsf{E}$. Show that $\mathsf{A} = \sigma\mathsf{A}(\mathsf{E})$.

2. Topology

Conventions

The set of rational numbers is $\mathbb{Q}$. If A and B are subsets of $\mathbb{R}$ then $A+B=\{x: x=a+b,\ a\in A,\ b\in B\}$. Similar conventions apply to AB and in general to "products" of subsets of algebraic structures. The set of complex numbers is $\mathbb{C}$ and $\mathbb{T}=\{z: z\in\mathbb{C},\ |z|=1\}$; the latter is regarded as a group under ordinary multiplication. The set of Borel sets of $\mathbb{R}$ is $\sigma\mathsf{R}(\mathsf{O}(\mathbb{R}))$ (see Problem 4).

If Γ and X are sets, X^Γ is the set of all maps of Γ into X. Equivalently, if $\Gamma=\{\gamma\}$ and if for all γ, $X_\gamma=X$ then X^Γ is the Cartesian product $\prod_\gamma X_\gamma$. (To reconcile these notations with 2^X as defined earlier in Set Algebra, Conventions, regard 2 as the set $\{0, 1\}$.) In particular, X^2, and more generally X^n, n in $\mathbb{N}$, is in some sense isomorphic to the n-factor Cartesian product of X with itself.

If X is a topological space containing A, A^0 is the interior of A (the union of all open sets contained in A), $\bar{A}$ is the closure of A (the intersection of all closed sets containing A). The set A is dense in X iff $\bar{A}=X$. If X and Y are topological spaces, $C(X, Y)$ is the set of continuous maps $f: X\mapsto Y$. A map $f: X^\Gamma\mapsto Y$ is finitely (countably) determined iff there is a finite (countable) set $\{\gamma_k\}$ in Γ so that if $x=\{x_\gamma\}$, $y=\{y_\gamma\}$, and $x_{\gamma_k}=y_{\gamma_k}$, $k=1, 2, \ldots$, then $f(x)=f(y)$. The expression $x_n\uparrow x$ $(x_n\downarrow x)$ means that the set $\{x_n\}_{n=1}^\infty$ is a monotone increasing (decreasing) sequence of real numbers converging to x. For a topological space X metrized by d, if $E\subset X$ then $\operatorname{diam}(E)=\sup\{d(x, y): x, y\in E\}$. The ball $\{x: d(x, y)\leqq r\}$ is $B(y, r)$ and for positive r, $B(y, r)^0$ is the open ball (both centered at y). If $x\in\mathbb{R}^n$ and $x=(x_1, x_2, \ldots, x_n)$ then $\|x\|=\sqrt{\sum_{k=1}^n x_k^2}$. This norm endows $\mathbb{R}^n$ with the standard Euclidean metric: $d:(x, y)\to\|x-y\|$. If $T:X\mapsto X$ is a map and n is in $\mathbb{N}$, T^n is the nth iterate of T, i.e., $T^n(x)=T(T^{n-1}(x))$, $n=2, 3, \ldots$.

If $\{X_\gamma\}_{\gamma\in\Gamma}$ is a set of topological spaces and $X=\prod_\gamma X_\gamma$, a basic neighborhood is defined by finitely many open sets U_{γ_i}, each contained in X_{γ_i}, $i=1, 2, \ldots, n$, and engendering the subset $U_{\gamma_1}\times U_{\gamma_2}\times\cdots\times U_{\gamma_n}\times\prod_{\gamma\notin\{\gamma_1,\gamma_2,\ldots,\gamma_n\}}X_\gamma$.

8. Prove or disprove: there is a continuous map of $[0, 1)$ onto $\mathbb{R}$.

9. Find in $[0, 1]$ an uncountable subset E such that the interior of $E-E$ is empty.

10. Metrize $[0, 1)$ so that it is complete and homeomorphic to $[0, 1)$ in its usual topology.

11. Show that if f maps $[0, 1]$ homeomorphically onto $A\times B$ (Cartesian product) then A or B consists of precisely one element.

12. Let *the* Cantor set C be $\{a: a=\sum_{k=1}^{\infty}\alpha_k 3^{-k}, \alpha_k=0 \text{ or } 2, k=1, 2, \ldots\}$. Let D_n be $\{0, 1\}$ in the discrete topology, $n=1, 2, \ldots$. Show C is homeomorphic to $\prod_{n=1}^{\infty}D_n$ and that $f_k: C\ni a\mapsto(-1)^{\alpha_k/2}$ is continuous, $k=1, 2, \ldots$.

13. Assume $\mathsf{F}(\mathbb{R})\ni F\subset\bigcup_{\gamma\in\Gamma}(a_\gamma, b_\gamma]$. Show there is in Γ a countable subset $\{\gamma_n\}$ such that $F\subset\bigcup_n(a_{\gamma_n}, b_{\gamma_n}]$.

14. Metrize $\mathbb{R}$ by $d:\mathbb{R}\times\mathbb{R}\ni(x, y)\mapsto|\arctan x-\arctan y|$. Prove or disprove that $\mathbb{R}$ is complete in the metric d.

15. Assume $S\subset[0, \infty)$, $u=\sup S$, $u<1$ and that if x and y are in S and $x<y$ then $x/y\in S$. Show that $u\in S$.

16. Prove or disprove: $\mathbb{R}\backslash\mathbb{Q}$ and $(\mathbb{R}\backslash\mathbb{Q})\cap(0, 1)$ are homeomorphic.

17. Let E be a compact, countable, and nonempty subset of $\mathbb{R}^2$. Show that there is an isolated point in E.

18. Show that if $\mathbb{R}^2$ is the countable union of closed sets F_n then the union of their interiors is dense in $\mathbb{R}^2$.

19. Show that if A is an uncountable subset of $\mathbb{R}^2$ then there is in A an x such that for every neighborhood $U(x)$, $U(x)\cap A$ is uncountable.

20. Construct a compact metrizable space X and a self-homeomorphism T of X (onto) X such that for no metric d compatible with the topology of X is it true that $d(Tx, Ty)=d(x, y)$ for all x and y in X.

21. Let X be a compact metric space and let $\{T_\gamma\}_{\gamma\in\Gamma}$ be an equicontinuous subset of $C(X, X)$. Show that for every f in $C(X, X)$ the set $\overline{\{f\circ T_\gamma\}_{\gamma\in\Gamma}}$ is compact.

22. Show that if X and Y are compact metric spaces and $f \in Y^X$, then f is continuous iff the graph $\{(x, f(x)): x \in X\}$ of f is closed in $X \times Y$.

23. Let X be a compact metric space with metric d and let f be in $C(X, X)$ and such that $d(f(a), f(b)) \geqq d(a, b)$ for all a and b in X. Show that $d(f(a), f(b)) = d(a, b)$ for all a, b in X.

24. Show that if X is a separable metric space then $\text{card}(\mathsf{F}(X)) \leqq \text{card}(\mathbb{R})$.

25. Let A be $C([0, 1], [0, 1])$ metrized according to $d: A \times A \ni (f, g) \mapsto \sup\{|f(x) - g(x)|: x \in [0, 1]\}$. Let A_i be the set of injective and A_s be the set of surjective elements of A and let A_{is} be $A_i \cap A_s$. Prove or disprove: i) A_i is closed; ii) A_s is closed; iii) A_{is} is closed; iv) A is connected; v) A is compact.

26. Let X be a complete metric space having no isolated points. Show that if U is a nonempty open set of X show then $\text{card}(U) \geqq \text{card}(\mathbb{R})$.

27. Let X and Y be compact Hausdorff spaces and $f: X \mapsto Y$ a continuous surjection such that for all y in Y, $f^{-1}(y)$ is connected. Show that for every connected subset C of Y, $f^{-1}(C)$ is connected. Give a counterexample to the conclusion if the hypothesis *Y is Hausdorff* is dropped.

28. Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a set of compact Hausdorff spaces. Show that if $X = \prod_\gamma X_\gamma$, $f \in C(X, \mathbb{R})$, and $\varepsilon > 0$ then there is a finitely determined $g: X \mapsto \mathbb{R}$ such that $\sup_x |f(x) - g(x)| < \varepsilon$. Show f is countably determined.

29. Let X be a compact space and let $f: X \mapsto \mathbb{R}$ be such that for all x, $f^{-1}([x, \infty))$ is closed. Show that for some M in $\mathbb{R}$ and for all x in X, $f(x) \leqq M$ and that for some x_0 in X, $f(x_0) = \sup_x f(x)$.

30. Let K be compact and a subset of the union of two open sets U and V in a Hausdorff space X. Show there are compact sets K_U and K_V contained respectively in U and V and such that $K = K_U \cup K_V$.

31. For all γ in Γ let I_γ be $[0, 1]$ and let X be $\prod_\gamma I_\gamma$. Show that if $\text{card}(\Gamma) = \text{card}(\mathbb{R})$ then there is a countable dense subset in X.

32. Show that $[0, 1]^{[0,1]}$ is not metrizable.

33. Show that if $f \in C([0, 1], \mathbb{R})$ and the subset A of $[0, 1]$ is the countable union of closed sets, i.e., A is an F_σ, then $f(A)$ is an F_σ.

34. Show that if $f \in C(\mathbb{T}, \mathbb{R})$ then there is in $\mathbb{T}$ a z such that $f(z) = f(ze^{i\pi})$. ("For some x in $\mathbb{R}$, $f^{-1}(x)$ contains two antipodal points.")

35. Show that if $f \in C(\mathbb{R}, \mathbb{R})$ and V is open then $f(V)$ is a Borel set.

36. If Y is a topological space such that for all n in $\mathbb{N}$, Y^n and Y are homeomorphic, need Y and $Y^{\mathbb{N}}$ be homeomorphic?

37. Let $\{F_n\}_{n=1}^\infty$ be a subset of $\mathsf{F}([0, 1])$. Show that if all F_n are nonempty and they are pairwise disjoint, then $[0, 1] \neq \cup_n F_n$.

3. Limits

Conventions

The series $\sum_{n=1}^{\infty} a_n$ may or may not converge. When it does its sum is $\sum_{n=1}^{\infty} a_n$. If p is a polynomial its degree is $\deg(p)$. The characteristic function of a set E is χ_E. If $n \geqq 2$ and if E is a Borel set in $\mathbb{R}^n$ the Lebesque measure of E is $\lambda_n(E)$. If $n = 1$, the corresponding number is $\lambda(E)$. If ambiguity is unlikely λ_n will be written λ.

38. Show there are real constants C and D such that if $n \geqq 2$ then $C \log n \leqq \sum_{k=1}^{\infty} (1-(1-2^{-k})^n) \leqq D \log n$.

39. Show that if $0 < a_n \leqq \sum_{k=n+1}^{\infty} a_k$, $n = 1, 2, \ldots$, and $\sum_{k=1}^{\infty} a_k = 1$, then for every x in $(0, 1)$ there is a subseries $\sum_{p=1}^{\infty} a_{k_p}$ whose sum is x.

40. Show that if a_n, $b_n \in \mathbb{R}$, $(a_n + b_n)b_n \neq 0$, $n = 1, 2, \ldots$, and both $\sum_{n=1}^{\infty} a_n/b_n$ and $\sum_{n=1}^{\infty} (a_n/b_n)^2$ converge, then $\sum_{n=1}^{\infty} a_n/(a_n+b_n)$ converges.

41. Find $\{a_n\}_{n=1}^{\infty}$ in $[0, \infty)$ so that $na_n \to 1$ as $n \to \infty$ and yet $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges.

42. Show that if $b_n \downarrow 0$ and $\sum_{n=1}^{\infty} b_n = \infty$ then there is in $\mathbb{R}$ a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n/b_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges.

43. Prove or disprove: If $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials for which $\deg(p_n) \leqq M < \infty$, n in $\mathbb{N}$, and $p_n \to f$ uniformly on $[0, 1]$ as $n \to \infty$ then f is a polynomial. Is pointwise convergence enough?

44. Show that $\int_x^{\infty} e^{-t^2/2}\, dt\, e^{x^2/2}$ is a monotone decreasing function of x on $[0, \infty)$ and that its limit as $x \to \infty$ is 0.

45. Show that $\lim_{n\to\infty} n \sin(2\pi e n!) = 2\pi$ (whence $e \notin \mathbb{Q}$).

46. Show that if $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}$ then $\lim_{n\to\infty} \int_0^{\infty} e^{-x}[\sin(x + r_n\pi/n)]^n \, dx = 0$.

47. Show $\lim_{\varepsilon\to 0} \int_0^{\infty} (1 - e^{(\varepsilon x)^2})\, e^{-x^3} \sin^4 x \, dx = 0$.

48. Evaluate: $\lim_{n\to\infty} \int_{-\infty}^{\infty} (1 - e^{-t^2/n})\, e^{-|t|} \sin^3 t \, dt$.

49. Let f be

$$[0,1] \ni x \mapsto \begin{cases} (x \log x)/(x-1) \text{ if } 0 < x < 1 \\ 0 \text{ if } x = 0. \\ 1 \text{ if } x = 1 \end{cases}$$

Show that $\int_0^1 f(x)\, dx = 1 - \sum_{n=2}^{\infty} 1/n^2(n-1)$.

4. Continuous Functions

Conventions

If A is a subset of a vector space V over a field $\mathbb{K}$ (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) the linear span of A is the set $\{\sum_{k=1}^{n} \alpha_k a_k : \alpha_k \in \mathbb{K},\ a_k \in A,\ n \in \mathbb{N}\}$ and the convex hull of A is the set $\{\sum_{k=1}^{n} \alpha_k a_k : \alpha_k \in \mathbb{R},\ 0 \leqq \alpha_k,\ \sum_{k=1}^{n} \alpha_k = 1,\ a_k \in A,\ n \in \mathbb{N}\}$. If X is a topological space and $f \in \mathbb{R}^X$ then for x in X, $\limsup_{y=x} f(y)$ ($\liminf_{y=x} f(y)$) is $\inf\{\sup_{y \in U} f(y) : U$ a neighborhood of $x\}$ ($\sup\{\inf_{y \in U} f(y) : U$ a neighborhood of $x\}$); f is upper (lower) semicontinuous, *usc* (*lsc*) iff $f(x) = \limsup_{y=x} f(y)$ ($\liminf_{y=x} f(y)$). The set $C_b(X, \mathbb{C})$ is the set of bounded continuous functions on X; its norm is given by $\|\cdot\cdot\cdot\|_\infty : f \mapsto \sup_x |f(x)|$.

The set $C^k(A, \mathbb{C})$, k in $\mathbb{N}$, consists of all functions having a kth derivative continuous on A; $C^\infty(A, \mathbb{C}) = \bigcap_{k=1}^{\infty} C^k(A, \mathbb{C})$; for k in $\mathbb{N}$ the norm $\|\cdot\cdot\cdot\|^{(k)}$ for $C^k([0, 1], \mathbb{C})$ maps f into $\sum_{j=0}^{k} \|f^{(j)}\|_\infty$. Similar definitions apply when $\mathbb{C}$ is replaced by $\mathbb{R}$. The unit ball of a Banach space is the set of elements with norm not greater than one.

To emphasize that an integration is carried out in the sense of Lebesgue rather than of Riemann the notation $\int_E f(x)\, d\lambda(x)$ will be used occasionally for the Lebesgue integral of f whereas $\int_E f(x)\, dx$ will be the only notation for the Riemann integral of f (in both cases over the set E). If X is a set, S is a σ-ring of sets in X, and μ is a measure defined on S then (X, S, μ) denotes the situation just described. If p is positive $L^p(X, \mu)$ is the set of (equivalence classes of) measurable functions f such that $\|f\|_p^p$ given by $\int_X |f(x)|^p\, d\mu(x)$ is finite; $L^\infty(X, \mu)$ is the set of (equivalence classes of) essentially bounded measurable functions f and $\|f\|_\infty$ is the essential supremum of $|f|$.

If X is a locally compact Hausdorff space, $C_0(X, \mathbb{C})$ is the set of continuous functions "vanishing at infinity", i.e., functions f such that for positive ε there is a compact set $K(\varepsilon, f)$ off of which $|f|$ is not more than ε; $C_{00}(X, \mathbb{C})$ is the set of continuous functions having compact support (supp(f), the support of f, is the closure of the set where f is not zero).

If f, g are $\mathbb{R}$-valued functions on a set X, $f^+ = (|f|+f)/2$, $f^- = (|f|-f)/2$; thus $g+(f-g)^+ = \max(f, g)$ and $g-(f-g)^- = \min(f, g)$, $f \vee g = \max(f, g)$, $f \wedge g = \min(f, g)$.

If E is a topological vector space over $\mathbb{K}$, E^* is the vector space of continuous linear maps of E onto $\mathbb{K}$; E^* is the conjugate or dual space of E.

50. Show that if $f, g \in C([0, 1], \mathbb{R})$ and $g(y_1) = g(y_2)$ whenever $f(y_1) = f(y_2)$ then there is a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that $p_n(f) \to g$ uniformly on $[0, 1]$ as $n \to \infty$.

51. Let f be defined as follows:

$$f(x) = \begin{cases} (3x+1)/2, & -1 \leqq x \leqq -1/3 \\ 0, & -1/3 < x \leqq 1/3 \\ (3x-1)/2, & 1/3 < x \leqq 1. \end{cases}$$

Let L belong to $C([-1, 1], \mathbb{C})^*$ and assume that if $n = 1, 2, \ldots$, $L(\underbrace{f \circ f \circ f \circ \cdots \circ f}_{n}) = 0$. Show that if $g \in C([-1, 1], \mathbb{C})$ and

$$g([-1/3, 1/3]) = \{0\}$$

then $L(g) = 0$.

52. For the maps $f_n: [0, 1] \ni x \mapsto e^{nx}$, n in $\mathbb{N}$, let S_N be $\{f_n : n \geqq N\}$, N in $\mathbb{N}$. Show that for all N in $\mathbb{N}$ the linear span of S_N is dense in $C([0, 1], \mathbb{C})$.

53. Show that if $f \in C([0, 1], \mathbb{R})$ and $\int_0^1 x^n f(x)\, dx = 0$ or $\int_0^1 e^{\pm 2\pi i n x} f(x)\, dx = 0$ for all n in $\mathbb{N} \cup \{0\}$ then $f = 0$.

54. Construct a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ so that for any f in $C([0, 1], \mathbb{C})$ and for which $f(0) = 0$ there is in $\mathbb{N}$ a sequence $\{n_k\}_{k=1}^{\infty}$ (dependent on f) such that $\sum_{n=1}^{n_k} a_n x^n \mapsto f$ uniformly on $[0, 1]$ as $k \to \infty$.

55. Show that if G is an open unbounded subset of $[0, \infty)$ and if $D = \{x : x \in (0, \infty), nx \in G$ for infinitely many n in $\mathbb{N}\}$ then D is dense in $[0, \infty)$.

56. Show that if $f \in C((0, \infty), \mathbb{R})$, $0 < a < b < \infty$, and for all h in (a, b), $f(nh) \to 0$ as $n \to \infty$ then $f(x) \to 0$ as $x \to \infty$.

57. Let f_0 be in $C([0, 1], \mathbb{R})$ and let $f_n(x)$ be $\int_0^x f_{n-1}(t)\, dt$ for n in $\mathbb{N}$ and x in $[0, 1]$. Show that if for each x in $[0, 1]$ there is in $\mathbb{N}$ an n (dependent on x) such that $f_n(x) = 0$, then the following are true: i) there is in $[0, 1]$ a nonempty open set on which f_0 is 0; ii) for every n in $\mathbb{N}$ and every b in $(0, 1]$, f_n has infinitely many zeros in $(0, b)$.

58. Let g be in $C([0,1],[0,1])$ and such that $g(0)=1-g(1)=0$. Let (*per convention*) g^n denote the n-fold composition $\underbrace{g\circ g\circ\cdots\circ g}_{n}$ and assume there is an m such that $g^m(x)=x$ for all x in $[0,1]$. Show $g(x)=x$ for all x in $[0,1]$.

59. Prove or disprove: i) if f is left-continuous on $[0,1]$ then f is bounded; ii) if f is *usc* on $[0,1]$ then f is bounded above.

60. Show that if $f\in C([0,1],\mathbb{R})$ and $f(0)=0$ then the sequence $\{\underbrace{f.f.\cdots.f}_{n}\}_{n=1}^{\infty}$ is equicontinuous iff $\|f\|_\infty<1$.

61. For f in $C([0,1],\mathbb{R})$ and n in $\mathbb{N}$ let a_n be $(\int_0^1 x^n f(x)\,dx)/(\int_0^1 x^n\,dx)$. Show that $\lim_{n\to\infty} a_n$ exists.

62. Let f be in $C([0,1],\mathbb{R})$ and assume that for some c in $(0,1)$ $\lim_{\substack{h\in\mathbb{Q},h\neq 0\\ h\to 0}}[f(c+h)-f(c)/h]$ exists and is L. Show f is differentiable at c.

63. Show that if $f,g\in C(\mathbb{R},\mathbb{R})$ and if, for all compactly supported h in $C^\infty(\mathbb{R},\mathbb{R})$, $\int_{-\infty}^{\infty} f(x)h(x)\,dx=-\int_{-\infty}^{\infty} g(x)h'(x)\,dx$, then g is differentiable and $g'=f$.

64. Let A be $\{f: f\in C^3([0,1],\mathbb{R}), \|f\|_\infty, \|f'''\|_\infty\leqq 1\}$. Show there is a constant K such that for all f in A, $\|f'\|_\infty, \|f''\|_\infty\leqq K$.

65. Assume $\{f_n\}_{n=1}^{\infty}\subset C([0,1],\mathbb{R})$, that each f_n is differentiable and that $\|f_n'\|_\infty\leqq 1$. Show that if $\int_0^1 f_n(x)g(x)\,dx\to 0$ as $n\to\infty$ for all g in $C([0,1],\mathbb{R})$ then $\|f_n\|_\infty\to 0$ as $n\to\infty$.

66. Prove or disprove: if $f\in C([1,\infty),\mathbb{R})$ there is a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that $p_n\to f$ uniformly on $[1,\infty)$ as $n\to\infty$.

67. Show that if $\{a_n\}_{n=1}^{\infty}\subset\mathbb{C}$ then there is in $C^\infty(\mathbb{R},\mathbb{C})$ an f such that for n in $\mathbb{N}$, $f^{(n)}(0)=a_n$.

68. Show that if $f\in C(\mathbb{R},\mathbb{R})$ and $|f|$ is improperly Riemann integrable then f is improperly Riemann integrable and Lebesgue integrable and that $\int_{\mathbb{R}} f(x)\,d\lambda(x)=\lim_{\substack{t\to\infty\\ s\to-\infty}}\int_s^t f(x)\,dx$.

69. Let f be in $C^1(\mathbb{R},\mathbb{C})$ and assume f' is real analytic (for each a in $\mathbb{R}$ there is a sequence $\{b_n(a)\}_{n=1}^{\infty}$ and a positive $r(a)$ such that for all x in $(a-r(a),a+r(a))f'(x)=\sum_{n=0}^{\infty} b_n(a)(x-a)^n$). Show f is real analytic.

70. Show that if $g\in C(\mathbb{R},\mathbb{R})$ and $\lim_{T\to\infty}\int_{-T}^{T} g(x)h(x)\,dx$ exists for all h in $L^2(\mathbb{R},\lambda)$, then $g\in L^2(\mathbb{R},\lambda)$.

71. Show that if $0<a, f\in C(\mathbb{R}^n,\mathbb{R}^n)$, and for all x, y in $\mathbb{R}^n$, $|f(x)-f(y)|\geqq a|x-y|$, then $f(\mathbb{R}^n)=\mathbb{R}^n$ (f is surjective).

72. Give a useful necessary and sufficient condition that a set F closed in $[0,\infty)$ be such that every f in $C([0,\infty),\mathbb{R})$ is uniformly approximable on