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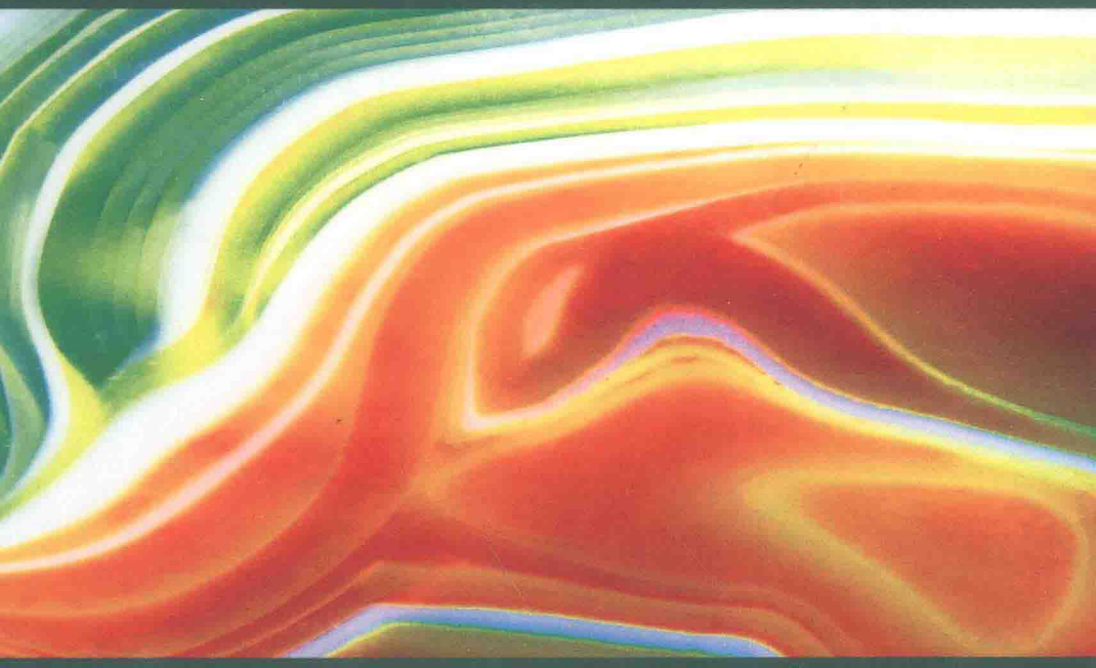

Homogenization of Heterogeneous Thin and Thick Plates

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Introduction

I.1. Motivation

Plate modeling is an old subject in mechanics, the main objective being to reduce the complexity of a three-dimensional (3D) model into a two-dimensional (2D) model without losing too much information about the 3D description of the fields. Depending on the plate slenderness and microstructure, numerous approaches were suggested. Whereas for a homogeneous plate, there are well established models, when the plate becomes a laminate made of highly anisotropic layers, the number of contributions is extremely large showing that some theoretical difficulties lie behind. When considering the very few contributions for modeling thick periodic plates, it appears clearly that there is a need for a well established method for deriving a plate model.

The motivation of this book is two-fold. First, in view of the broad and eclectic literature regarding thick plate models, it seems an interesting challenge to suggest an approach which enables the derivation of a thick plate model which is efficient for homogeneous plates, laminated plates and also periodic plates. Second, it puts a new perspective on the original work of Reissner [REI 45] which used the minimum of complementary energy for deriving a thick plate model in the isotropic and homogeneous case. From this approach, the thin plate model (Kirchhoff–Love) may be retrieved without inconsistencies often encountered in axiomatic derivations. Whereas the original derivation from [REI 45] was for homogeneous plates, it is possible to extend it soundly to the case of monoclinic laminated plates, the price being the introduction of a generalized shear force which has 6 static degrees

of freedom (d.o.f.) instead of the 2 d.o.f. conventional shear force. This theory is called the Bending-Gradient. It is also extended to periodic plates which are a challenge. Finally, this book is also an opportunity to show relevant applications of the Bending-Gradient theory.

Since the main objective of this book is to derive a new plate theory, it should be understood that a minimum knowledge of continuum mechanics and classical plate models is necessary. Hence, the expected audience begins at Master's students. Nevertheless, for self-consistency, linear elasticity and most basic plate model are covered. Additionally, all mathematical developments are formal, in the sense that underlying Sobolev spaces and topology are not specified and no convergence results are sought. However, the derivation is performed with as much care as possible so that rigorous proofs may be accessible.

I.2. A brief history of plate models

The number of contributions regarding plate models is so large that it is an impossible task to provide an exhaustive review. We attempt here to perform a general history of plate models which also corresponds to the organization of this book, starting with the simplest models (the homogeneous and isotropic plate) to the more elaborated models (periodic plates).

The approaches for deriving a plate model may be separated in two main categories: axiomatic and asymptotic approaches. Axiomatic approaches start with *ad hoc* assumptions on the 3D field representation of the plate, separating the out-of-plane coordinate from the in-plane coordinates. Most of the time, it is the 3D displacement distribution which is postulated and the minimum of potential energy is invoked for deriving a plate model. The limitation of these approaches comes from the educated guess for the 3D field distribution which is often specific to the plate microstructure. Asymptotic approaches often come after axiomatic approaches. They are based on the introduction of a scaling parameter which is assumed to go to 0 in the equations of the 3D problem. In the case of plate models, it is the thickness t divided by the span L (the inverse of the slenderness) which is assumed to be extremely small. Following a rather well established procedure (asymptotic expansions, Γ-convergence), they enable the derivation of plate models (often justifying *a posteriori* axiomatic approaches) and are the basis of a

convergence result. Their only limitation is that the plate does not often exactly follow the asymptotic assumption: it may not be so slender or have high contrast ratios in its constitutive materials. Hence, having the fastest convergence rate may not lead to an accurate model in practical situations.

I.2.1. *Thin plate theories for homogeneous and laminated plates*

The very first attempt to derive a plate model in bending came from Sophie Germain. In 1809, the Paris Academy of Sciences sponsored a contest related to an experiment of Ernst Chladni. The latter excited metal plates and observed mode-shapes. The objective of the contest was to suggest a model supporting this observation. Sophie Germain managed to obtain, for the first time, the equation of motion of a thin and homogeneous plate (though the derivation itself was incorrect). Later, Kirchhoff suggested making the assumption that the transverse displacement of the plate was uniform through the thickness and that the normal line to the midsurface of the plate remained normal through the transformation [KIR 50]. These assumptions enabled Love [LOV 88] to correctly establish a thin plate theory for homogeneous and isotropic plates which is often referred to as Kirchhoff–Love plate theory. However, the axiomatic derivation of this theory suffers from a contradiction. Assuming the transverse displacement is uniform through the thickness means that the out-of-plane strain is zero and leads to a plane-*strain* constitutive equation in bending. This contradicts the natural scaling of the stresses in the plate which shows that the normal stress must vanish at leading order and would rather lead to a plane-*stress* constitutive equation in bending. This contradiction was resolved by an asymptotic derivation of the Kirchhoff–Love theory [CIA 79]. It appeared that the equations derived by Love are indeed the leading order of the asymptotic expansion and that Kirchhoff assumptions are correct at leading order for the displacement field. However, the strain field directly derived from this displacement is incorrect because it misses the higher order contribution from transverse Poisson's effect. The convergence of Kirchhoff–Love plate model was studied in detail since the pioneering work from [MOR 59]. It was established that when the plate is clamped on the boundary (all the 3D displacement is restrained) the error estimate converges as $(t/L)^{1/2}$, where t is the thickness and L the span of the plate, whereas if the plate has only simply supported or free boundaries the convergence rate is $(t/L)^2$ (see [CIA 97]). The rather poor convergence

rate when the plate is clamped comes precisely from the restrained transverse Poisson's effect on the boundary. This generates a boundary layer which cannot be captured by Kirchhoff–Love plate theory.

Laminated plates are made of a succession of homogeneous layers of elastic material. The constitutive material of the layers is often highly anisotropic with different orientations in each layers. A typical illustration is plywood or laminated plates made of carbon fibers reinforced polymers layers. The axiomatic approach from [LOV 88] was applied quite early to plywood [MAR 36][1] and is often referred today as Classical Lamination Theory. The leading order of the asymptotic expansion is a straightforward extension of the homogeneous case (see [LEB 13b] for instance) and similar convergence results may be found resolving again the inconsistencies of the assumed kinematics.

In the present book, the Kirchhoff–Love plate theory is derived by application of the minimum of complementary energy after derivation of a statically compatible stress field instead of postulating the kinematics in Chapter 2. This derivation does not require explicitly asymptotic expansions and avoids also the inconsistencies coming from Kirchhoff assumptions.

I.2.2. *Thick plate theories for homogeneous and laminated plates*

In the Kirchhoff–Love plate model, the transverse shear stress energy is neglected because it is related to higher order effects with respect to the slenderness of the plate. However, there are many practical cases where this approximation is too coarse. First, plates are not really slender in applications. This is especially true in civil engineering where rather large loads must be carried by floors which usually sets the slenderness between $L/t = 10$ and $L/t = 30$. Second, depending on the plate microstructure, high anisotropy may be encountered which possibly increases the contribution of the shear energy. Typical examples are with sandwich panels and laminated plates. Sandwich panels include a very compliant core layer which is "sandwiched" between two rather stiff skins. Laminated plate may show large contrast between the Young modulus in the fiber's direction and the transverse shear modulus across the fibers. With these kinds of plates, the deflection predicted

1 It is the earliest reference known by the authors.

by the Kirchhoff–Love model may be rather inaccurate. Additionally, the knowledge of the actual transverse shear stress distribution in the plate is not provided by the Kirchhoff–Love plate model whereas it is a critical piece of information for the engineer in order to predict the failure.

Several attempts to derive a thick plate model for homogeneous and isotropic plates were published in a short time interval by [REI 45, HEN 47, BOL 47]. These approaches apparently led to the same macroscopic equations. However, because their derivation is based on different assumptions, the mechanical meaning of the plate variables is not exactly the same.

Reissner assumed a stress distribution related to bending linearly distributed through the thickness and derived a statically compatible stress field [REI 45, REI 47]. More precisely, the transverse shear distribution was parabolic through the thickness and proportional to the shear force. Applying the minimum of complementary energy to this distribution drove him to a thick plate model in which the kinematic variables are the deflection and two rotations fields. These plate generalized displacements were defined as weighted averages of 3D displacements. The strength of this approach is that it provides a good 3D estimate of the stress, as well as deflection, in the plate. It also avoids the kinematic inconsistencies encountered with Kirchhoff–Love model. However, the definition of generalized displacements being indirectly related to the 3D displacement was not very practical. Reissner himself introduced mixed variational principles in order to resolve this difficulty [REI 50].

Exactly the same as for the Timoshenko beam model, where the rotation of the section is an independent variable, Hencky [HEN 47] and Bollé [BOL 47] assumed that the normal to the plate in not restrained to remain normal to the mid-surface through the transformation. This introduced again two independent rotation fields directly related to the 3D the kinematics. This approach, referred to as first-order shear deformation theory (FOSDT), leads to a uniform transverse shear strain through the thickness related to the difference between the slope of the mid-plane of the plate and the actual inclination of the material normal line. This strain distribution leads also to a uniform transverse shear stress through the thickness which does not satisfy lower and upper free boundary of the plate and underestimates the actual maximum shear stress contrary to Reissner's approach. Again, a too crude

axiomatic kinematics leads to overestimating the actual stiffness of the plate both in bending and also in transverse shear.

The first workaround to Hencky's kinematics was the introduction of shear correction factors in order to take into account the actually non-uniform distribution of the transverse shear stress as had already been done with Timoshenko beam model. Because the contribution from Reissner [REI 45] led to the same equations as those of Hencky except that the shear constitutive equation was multiplied by $5/6$, this value was considered as a good estimate for the shear correction factor. Shortly after, Mindlin [MIN 51] suggested another value ($\pi^2/12$) for the shear correction factor, based on dynamic considerations. Beyond the question of which is the correct factor, changing the shear constitutive equation will not enable the derivation of better estimates of the transverse shear distribution.

The limitations of shear correction factors encouraged the exploration of enriched kinematics. A fairly large amount of suggestions were made (see [LEV 80, RED 84, TOU 91] for instance) and an interesting discussion about the connections between them is provided by [LEW 87]. These models give rather good estimates of the transverse shear stress distribution and the deflection. This idea was pushed further with hierarchical models (see the digest from [DAU 00]) where the 3D displacement is assumed as a polynomial of the transverse coordinate and each monomial is multiplied by in-plane function being a generalized plate displacement. In case the plate is simply supported, it is possible to prove a higher-order convergence rate with respect to the plate slenderness. However, when the plate is clamped, it is actually not possible to improve the sharp bound observed with Kirchhoff–Love plate model.

Higher order asymptotic expansions were also performed [DAU 95] and higher order convergence results established. However, these improved estimates require boundary layer terms and the solution of embedded Kirchhoff–Love problems which are impractical for engineers.

In the end, the most widely implemented plate model is still the 3 kinematic degrees of freedom Reissner–Mindlin model (one deflection two rotations). There are several reasons for this. First, it requires few d.o.f. with only first-order derivatives in the constitutive equation. Second, the boundary conditions have a simple mechanical meaning. It has thus become an endeavor to extend

this model to more complex plate microstructures such as laminated plates. Note that, whereas Reissner and, almost simultaneously, Hencky were the first to suggest this model from different assumptions, the denomination Reissner–Mindlin is more common and we tend to this use when referring to the plate equations.

Whereas an educated guess was still rather easy for a homogeneous plate, finding a kinematics which captures correctly the effects of transverse shear strain is much more difficult with laminated plates. It turns out that most of the efforts for modeling this kind of plates were turned in this direction, leading to a vast literature of refined models [RED 89, ALT 98, NOO 00, CAR 02].

Applying FOSDT directly to a laminated plate leads to a discontinuous transverse shear stress distribution and incorrect estimation of the actual deflection compared to exact solutions. Now, the definition of a shear correction factor becomes meaningless since there may be different constitutive materials (see the illustrative discussion for a sandwich panel in [BIR 02]). Nevertheless assuming the plate is under cylindrical bending, Whitney [WHI 72] suggested a derivation of shear correction factors. However, there is no reason to expect these corrections being valid in more general configurations.

Enriched kinematics are mostly based on a generalization of an idea from [AMB 69] which allows the derivation of a transverse shear stress which is continuous ([RED 84, TOU 91] among many other suggestions). However, these approaches do not lead to a Reissner–Mindlin plate model and are still based on axiomatic arguments.

Because of the difficulties encountered with the description of transverse shear stress in laminates, layerwise approaches have also been investigated [CAR 02, DIA 01]. In these approaches, each layer of the laminate is treated as an individual plate with its own generalized plate variables. This enables a very accurate description of the 3D fields especially close to the boundary where stress singularity occurs leading to inter-laminar failures [SAE 12b, SAE 12a]. The main limitation of these approaches is that they require a large and varying number of d.o.f.

Finally, asymptotic approaches were also applied to laminated plates however, going higher order does not lead to a Reissner–Mindlin model [SUT 96, YU 02].

In Chapter 4, the original derivation from Reissner of a thick plate model for the homogeneous case is recalled in detail. Since this derivation is based on the minimum of complementary energy, the Reissner model is an upper bound of the 3D external work and consequently of the displacement. In addition to the original derivation, a 3D displacement localization (often called "displacement recovery") in agreement with this bound is suggested. Then, the application of exactly the same procedure to the case of a laminated plate is presented. This approach requires the introduction of the first and second gradient of the bending moment as generalized static variables and leads to a model called "Generalized Reissner" [LEB 15]. This model involves 15 kinematic d.o.f., most of them related to out-of-plane Poisson's distortion, not really relevant for practical applications. However, it complies rigorously with the minimum of complementary energy, allows a clear definition of plate generalized displacement as well as 3D displacement localization. Finally, in order to derive a more practical model, a kinematic assumption (locking Poisson's distortion) leads to a simpler model called the Bending-Gradient theory, formally closer to the simpler Reissner–Mindlin model. The Bending-Gradient theory replaces the classical shear force by a generalized shear force related to the first gradient of the bending moment. Depending on the plate microstructure, this model may be turned into the original Reissner–Mindlin model. This is typically the case when the plate is homogeneous. Finally, this new plate theory is seen by the authors as an extension of Reissner's theory to heterogeneous plates which preserves most of its simplicity.

Application of the Bending-Gradient theory to laminates made of carbon fiber reinforced polymers in Chapter 7 shows that the Bending-Gradient theory does not increase the convergence rate of Kirchhoff–Love theory in terms of deflection. However, its prediction compared to a reference solution are considerably better (about two order of magnitude). Moreover, the convergence rate of the error in terms of 3D stress field is larger once the transverse shear distribution is taken into account.

I.2.3. *Periodic plates*

A periodic plate is a flat object made of the repetition in its plane of a single unit-cell. For instance, honeycomb sandwich panels and corrugated cardboard include a periodic core (a honeycomb or a flute). Other examples

are reticulated space-frame, beam lattice, concrete waffle slabs, etc. Seen from far these objects may be considered as a plate in order to reduce the computation burden.

Axiomatic approaches, where a 3D kinematics is based on an educated guess, are much more difficult, if not impossible, to apply, with periodic plates because of the wide diversity of unit-cells. Hence, most suggestions are based on asymptotic approaches and are related to homogenization techniques.

It turns out that performing asymptotic expansions on a periodic plate where the size of the unit-cell becomes small with respect to the span of the plate leads to a Kirchhoff–Love plate model [CAI 84, KOH 84]. The overall picture is that instead of explicitly enforcing Kirchhoff–Love kinematics everywhere on the unit-cell, this kinematics must be applied *on average* to the unit-cell. This result unifies the already known results for homogeneous and laminated plates and provides a simple mechanical interpretation of the effect of bending in periodic plates.

Exactly the same as for laminated plates, the question of the effect of shear forces and the related deflection is of great interest for engineers. Very few suggestions are present in the literature. Lewinski [LEW 91a, LEW 91b, LEW 91c] performed the asymptotic expansion up to second-order for periodic plates. However, no plate theory was derived.

In Part 3 of this book, the homogenization scheme for thin plate from [CAI 84, KOH 84] is given a new perspective. Then, following the same approach as with laminated plates, a homogenization scheme leading to a Bending-Gradient plate model is derived. This enables the application to sandwich panels and a simple beam lattice. In Chapter 10, it is shown that under the contrast assumption usually made between the skins and the core of a sandwich panel, it is possible to consider such periodic plates as a Reissner–Mindlin plate. The corresponding homogenization scheme is fully detailed. Finally, considering a beam lattice is an opportunity to show that there are some plates which may never be turned into a Reissner–Mindlin plate. The very simple lattice which is under consideration allows clear illustrations of the effects of the bending moment and also the generalized shear force which is the new static unknown introduced by the Bending-Gradient theory.

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