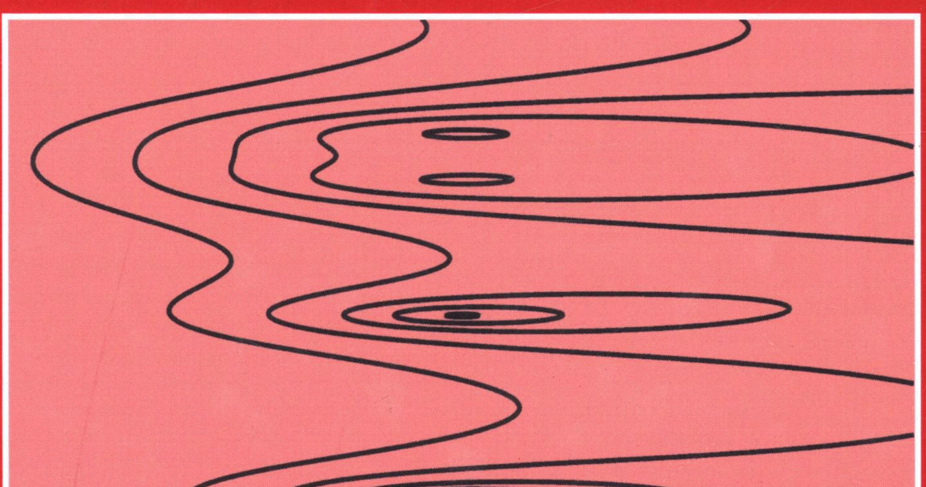
Encyclopedia of Mathematics and Its Applications 165

EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume Two: Analytic Equivalents

Kevin Broughan



The Riemann hypothesis (RH) is perhaps the most important outstanding problem in mathematics. This two-volume text presents the main known equivalents to RH using analytic and computational methods. The books are gentle on the reader with definitions repeated, proofs split into logical sections, and graphical descriptions of the relations between different results. They also include extensive tables, supplementary computational tools, and open problems suitable for research. Accompanying software is free to download.

These books will interest mathematicians who wish to update their knowledge, graduate and senior undergraduate students seeking accessible research problems in number theory, and others who want to explore and extend results computationally. Each volume can be read independently.

Volume 1 presents classical and modern arithmetic equivalents to RH, with some analytic methods. Volume 2 covers equivalences with a strong analytic orientation, supported by an extensive set of appendices containing fully developed proofs.

Kevin Broughan is Emeritus Professor in the Department of Mathematics and Statistics at the University of Waikato, New Zealand. In these two volumes he has used a unique combination of mathematical knowledge and skills. Following the publication of his Columbia University thesis, he worked on problems in topology before undertaking work on symbolic computation, leading to development of the software system SENAC. This led to a symbolic-numeric dynamical systems study of the zeta function, giving new insights into its behaviour, and was accompanied by publication of the software GL(n) pack as part of D. Goldfeld's book, *Automorphic Forms and L-Functions for the Group GL(*n,*R)*. Professor Broughan has published widely on problems in prime number theory. His other achievements include co-establishing the New Zealand Mathematical Society, the School of Computing and Mathematical Sciences at the University of Waikato, and the basis for New Zealand's connection to the internet.

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-107-19712-1

9 781107 197121 >

165

Broughan

EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume 2

CAMBRIDGE

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Equivalents of the Riemann Hypothesis

Volume Two: Analytic Equivalents

KEVIN BROUGHAN

University of Waikato, New Zealand

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107197121
DOI: 10.1017/9781108178266

First published 2017

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloguing in Publication Data
Names: Broughan, Kevin A. (Kevin Alfred), 1943– author.
Title: Equivalents of the Riemann hypothesis / Kevin Broughan,
University of Waikato, New Zealand.
Description: Cambridge : Cambridge University Press, 2017– |
Series: Encyclopedia of mathematics and its applications ; 165 |
Includes bibliographical references and index. Contents: volume 2. Analytic Equivalents
Identifiers: LCCN 2017034308 | ISBN 9781107197121 (hardback : alk. paper : v. 1)
Subjects: LCSH: Riemann hypothesis.
Classification: LCC QA246 .B745 2017 | DDC 512.7/3–dc23
LC record available at https://lccn.loc.gov/2017034308

ISBN – 2 Volume Set 978-1-108-29078-4 Hardback
ISBN – Volume 1 978-1-107-19704-6 Hardback
ISBN – Volume 2 978-1-107-19712-1 Hardback

EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume Two: Analytic Equivalents

The Riemann hypothesis (RH) is perhaps the most important outstanding problem in mathematics. This two-volume text presents the main known equivalents to RH using analytic and computational methods. The books are gentle on the reader with definitions repeated, proofs split into logical sections, and graphical descriptions of the relations between different results. They also include extensive tables, supplementary computational tools, and open problems suitable for research. Accompanying software is free to download.

These books will interest mathematicians who wish to update their knowledge, graduate and senior undergraduate students seeking accessible research problems in number theory, and others who want to explore and extend results computationally. Each volume can be read independently.

Volume 1 presents classical and modern arithmetic equivalents to RH, with some analytic methods. Volume 2 covers equivalences with a strong analytic orientation, supported by an extensive set of appendices containing fully developed proofs.

Encyclopedia of Mathematics and Its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit
www.cambridge.org/mathematics.

119 M. Deza and M. Dutour Sikirić *Geometry of Chemical Graphs*
120 T. Nishiura *Absolute Measurable Spaces*
121 M. Prest *Purity, Spectra and Localisation*
122 S. Khrushchev *Orthogonal Polynomials and Continued Fractions*
123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivity*
124 F. W. King *Hilbert Transforms I*
125 F. W. King *Hilbert Transforms II*
126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
127 M. Grabisch *et al. Aggregation Functions*
128 L. W. Beineke and R. J. Wilson (eds.) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
129 J. Berstel, D. Perrin and C. Reutenauer *Codes and Automata*
130 T. G. Faticoni *Modules over Endomorphism Rings*
131 H. Morimoto *Stochastic Control and Mathematical Modeling*
132 G. Schmidt *Relational Mathematics*
133 P. Kornerup and D. W. Matula *Finite Precision Number Systems and Arithmetic*
134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
136 A. Kristály, V. D. Rădulescu and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein *et al. Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn–Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. Molica Bisci, V. Radulescu and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 S. Wagon *The Banach–Tarski Paradox (Second Edition)*
164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
166 M. Baake and U. Grimm *Aperiodic Order II: Representation Theory and the Zelmanov Approach*

Dedicated to Jackie, Jude and Beck

RH is a precise statement, and in one sense what it means is clear, but what it is connected with, what it implies, where it comes from, can be very unobvious.

Martin Huxley

Preface

Why have these two volumes on equivalences to the Riemann hypothesis been written? Many would say that the Riemann hypothesis (RH) is the most noteworthy problem in all of mathematics. This is not only because of its relationship to the distribution of prime numbers, the fundamental building blocks of arithmetic, but also because there exist a host of related conjectures that will be resolved if RH is proved to be true and which will be proved to be false if the converse is demonstrated. These are the RH equivalences. The lists of equivalent conjectures have continued to grow ever since the hypothesis was first enunciated, over 150 years ago.

The many attacks on RH that have been reported, the numerous failed attempts, and the efforts of the many whose work has remained obscure, have underlined the problem's singular nature.

The aim of these volumes is to give graduate students and number theory researchers easy access to these methods and results in order that they might build on them. To this end, complete proofs have been included wherever possible, so readers might judge for themselves their depth and crucial steps. In a few places the more philosophical thoughts of experts have been reported. These for the most part have been paraphrased or quoted from the books of du Sautoy [215] or Sabbagh [210].

The two volumes are distinct, with a small amount of overlap. The first, Volume One [39], has an arithmetic orientation, with some analytic methods, especially those relying on the manipulation of inequalities. The equivalences found there are those of Caveney–Nicolas–Sondow, Franel–Landau, Hilbert–Pólya, Lagarias, Littlewood, Landau, Nicolas, Nazardonyavi–Yakubovich, Ramanujan–Robin, Redheffer, Shapiro, Shoenfeld, Spira and Shapiro. In addition, Volume One has criteria based on the divisibility graph, Dirichlet eta function and the symmetric group. There is a supporting *Mathematica*™ package, RHpack.

Volume Two, this book, contains equivalences with a strong analytic orientation. To support these, there is an extensive set of appendices containing fully developed proofs of most results. The equivalences set out are the series criteria of Riesz, Hardy–Littlewood and Báez-Duarte, the L_p space condition of Beurling, the Sondow–Dumitrescu criterion based on the monotonicity of $|\xi(s)|$, the inequality criterion of Li and its extension by Lagarias and Bombieri, the de Bruijn–Newman constant criterion, the orthogonal polynomial criterion of Cardon–Roberts, the cyclotomic polynomial criterion of Amoroso, the integral equations of Sekatskii–Beltraminelli–Merlini, Salem and Levinson, the explicit-formula-based inequality of Weil, the variational criterion of Bombieri, the discrete measures of Verjovsky, the horocycle-measure-based criterion of Zagier, the Hermitian forms of Yoshida, and smooth integer estimate ranges of Hildebrand. In addition, Bombieri's proof of Weil's explicit formula is given, as is a discussion of the Weil conjectures and a proof of the conjectures in the case of elliptic curves.

In the case of the general Riemann hypothesis (GRH) for Dirichlet L-functions, the Titchmarsh criterion is given, as well as proofs of the Bombieri–Vinogradov and Gallagher theorems and a range of their applications. There is a small supporting *Mathematica* package, GRHpack, with access details below.

To aid the reader, definitions are often repeated and major steps in proofs are numbered to give a clear indication of the main parts and allow for easy proof internal referencing. When possible, errors in the literature have been corrected. Where a proof has not been verified, either because this author was not able to fill gaps in the argument, or because it was incorrect, it has not been included. There is a website for errata and corrigenda, and readers are encouraged to communicate with the author in this regard at kab@waikato.ac.nz. The website is linked to the author's homepage: www.math.waikato.ac.nz/~kab.

Also linked to this website is the suite of *Mathematica* programs, called GRHpack, related to the material in this volume, which is available for free download. Instructions on how to download the software are given in Appendix L.

Many people have assisted with the development and production of these volumes. Without their help and support, the work would not have been possible, and certainly not completed in a reasonable period of time. They include Sir Michael Berry, Enrico Bombieri, Jude Broughan, George Csordas, Daniel Delbourgo, Tomás Garcia Ferrari, Pat Gallagher, Adolf Hildebrand, Geoff Holmes, Stephen Joe, Jeff Lagarias, Wayne Smith, Tim Trudgian, John Turner and Michael Wilson. The support of the University of Waikato and especially its Faculty of Computing and Mathematical Sciences and Department of Mathematics and Statistics has been absolutely essential.

Cambridge University Press has also provided much encouragement and support, especially Roger Astley and Clare Dennison. Last, but not least, I am grateful for my family's belief in me and support of my work.

Kevin Broughan
December 2016

Acknowledgements

The author gratefully acknowledges the following sources and/or permissions for the non-exclusive use of copyrighted material.

G. H. Hardy: Figure 2.1, Mondadori Portfolio/Getty Images.

A. Beurling: Figure 3.1, photograph by Anne-Marie Xykull Gyllenband, permission of Institut Mittag-Leffler of the Royal Swedish Academy of Science.

E. Bombieri: Figure 4.2, Herman Landshoff photographer. From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA.

G. Pólya: Figure 5.1, by Marion Walter, 1976.
www.cah.utexas.edu/collections/math_walter_gallery.php
New Mathematical Library Records, Archives of American Mat e_math_00414, The Dolph Briscoe Center for American History, The University of Texas at Austin.

N. de Bruijn: Figure 5.2, Author: Konrad Jacobs. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

D. Cardon: Figure 6.1, used by permission of D. Cardon.

A. Weil and A. Selberg: Figure 9.1, Author: Konrad Jacobs. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

Weil's commentary: Section 9.16, permission of Tia An Wong to use his translation of Weil's commentary on his 1952 and 1974 papers.

H. Yoshida: Figure 11.1, used by permission of H. Yoshida.

E. C. Titchmarsh: Figure 12.1, University of Oxford Mathematics Institute, with the permission of Jennifer Andrews née Titchmarsh.

A. Hildebrand: Figure 13.1, used by permission of A. Hildebrand.

Contents for Volume Two