Springer Texts in Business and Economics

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Mathematical Financial Economics

A Basic Introduction

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ISSN 2192-4333 ISSN 2192-4341 (electronic)
Springer Texts in Business and Economics
ISBN 978-3-319-16570-7 ISBN 978-3-319-16571-4 (eBook)
DOI 10.1007/978-3-319-16571-4

Library of Congress Control Number: 2015939236

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

More information about this series at
http://www.springer.com/series/10099

Preface

Tyche
Goddess of Chance and Fortune
By Tatjana Heinz

This textbook is a basic introduction to the key topics in mathematical finance and financial economics—two realms of ideas that substantially overlap but are often treated separately from each other. Our goal is to present the highlights in the field, with the emphasis on the financial and economic content of the models, concepts and results. The book provides a novel, unified treatment of the subject by deriving each topic from common fundamental principles and showing the interrelations between the key themes.

Although our presentation is fully rigorous, with some rare and clearly marked exceptions, we restrict ourselves to the use of only elementary mathematical concepts and techniques. No advanced mathematics (such as stochastic calculus) is used. The main source for the book, and a "proving ground" for testing our presentation of the material, are courses on mathematical finance, financial economics and risk management which we have delivered, over the last decade, to undergraduate and graduate students in economics and finance at the Universities of Manchester, Zurich and Leeds.

The textbook contains 18 chapters corresponding to 18 lectures in a course based upon it. There are three chapters with problems and exercises, most of which have been used in tutorials, take-home tests and examinations, with full and detailed answers. The problems and exercises contain not only numerical examples, but also theoretical questions that complement the material presented in the body of the textbook. Two mathematical appendices provide rigorous definitions of some of the mathematical notions and statements of general theorems used in the text.

The textbook covers the classical topics, such as mean-variance portfolio analysis (Markowitz, CAPM, factor models, the Ross-Huberman APT), derivative securities pricing, and general equilibrium models of asset markets (Arrow, Debreu and Radner). A less standard but very important topic, which to our knowledge has not previously been covered in elementary textbooks, is capital growth theory (Kelly, Breiman, Cover and others). Absolutely new material, reflecting research achievements of recent years, is an introduction to new dynamic equilibrium models of financial markets combining behavioral and evolutionary principles.

A characteristic feature of financial economics is that it has to focus on the analysis of random, unpredictable market situations. To model such situations our discipline created powerful theoretical tools based on probability and stochastic processes. However, the power of human mind is not unlimited, and it can never fully eliminate the influence of chance and fortune, personified by goddess Tyche, looking at us from the epigraph to this book.

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1.1 Asset Prices and Returns

Assets We will consider a financial market where N *assets* (*securities*) $i = 1, 2, \ldots, N$ are traded. Typical assets are common stocks, bonds, domestic or foreign cash, etc. Generally, the term "asset" is associated with any financial instrument that can be bought or sold.

Return Each asset is characterized by its *return* R_i. The return R_i on asset i is a random variable. For the purposes of mathematical modelling, we will assume that some characteristics of the random variables R_i, $i = 1, 2, \ldots, N$, (e.g. expectations and covariances) are known.

Asset Prices and Returns How are asset returns computed? We consider a model in which there are two moments of time 0 and 1. Let $S_0^i > 0$ be the price of asset i at time 0 and $S_1^i \geq 0$ the price of the asset at time 1. Then the *asset return* can be defined as

$$R_i = \frac{S_1^i - S_0^i}{S_0^i}.$$

This expression is also termed *the rate of return*.

Vectors of Prices and Vectors of Returns The financial market under consideration is specified by a random vector

$$R = (R_1, \ldots, R_N),$$

I.V. Evstigneev et al., *Mathematical Financial Economics*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-16571-4_1

whose ith component R_i represents the return on asset i. In what follows, we will denote by S_0 and S_1 the price vectors

$$S_0 = (S_0^1, \ldots, S_0^N), \; S_1 = (S_1^1, \ldots, S_1^N).$$

The price vector S_1 is random (not known at time 0, but known at time 1), while S_0 is fixed (known at time 0).

1.2 Investor's Portfolio: Long and Short Positions

Investor's Portfolio The problem of an investor is to decide what amount of what asset to buy, or in other words, what *portfolio* of assets to select. A portfolio x can be characterized by a vector

$$x = (x_1, \ldots, x_N),$$

where x_i denotes the amount of money invested in asset i. Assets are purchased at time 0, when their prices are $S_0^1, \ldots, S_0^N$. Total wealth invested in the portfolio $x = (x_1, \ldots, x_N)$ at time 0 is

$$w_0 = x_1 + \ldots + x_N.$$

For each i, the ratio

$$h_i = \frac{x_i}{S_0^i}$$

is the number of ("physical") units of asset i in the portfolio $x = (x_1, \ldots, x_N)$. At time 0, one can equivalently specify a portfolio in terms of the vector $(h_1, \ldots, h_N)$ or in terms of the vector $(x_1, \ldots, x_N)$.

Long and Short Positions of a Portfolio *Positions* of a portfolio (expressed in terms of money or in terms of physical units of assets) are the coordinates of the corresponding vector $x = (x_1, \ldots, x_N)$ or $h = (h_1, \ldots, h_N)$. These coordinates generally might be positive (*long* positions) or negative (*short* positions). A positive coordinate x_i = €100 of the vector x means that the investor owns the amount of asset i that costs (at time 0) €100. If, for example, S_0^i = €20, then the fact that x_i = €100 means that the investor owns $x_i / S_0^i = 100/20 = 5$ units of asset i.

What Does a Negative Coordinate x_i Mean? Negative positions of a portfolio appear, in particular, in the following three cases.

(a) If one of the assets, say $i = 1$, is cash, the negative number $x_1 = -€10{,}000$ means that the investor has *borrowed* €10,000 (for example, from a bank) at time 0. It is supposed that at time 1, the investor has to pay the debt $€(1+r)10{,}000$, where $r \geq 0$ is the interest rate.
(b) Negative coordinates x_i of the portfolio might reflect a possibility of *short selling*. An investor may be allowed to borrow some amount of asset i from somebody (say, a broker) and sell this amount on the market at the prevailing price. At a later date, however, the assets must be returned. This may lead either to gains or to losses, depending on whether the price of the asset has decreased or increased during the time period under consideration. Short selling is a risky operation, and, in real financial institutions, it is often prohibited, or at least restricted by certain regulations.
(c) Suppose asset i is given by a contract, typically these are of some standardized form (an important example is an *option*; such contracts will be considered in detail later). Contracts of this kind may be written, sold and purchased by market traders. The one who writes contract i at time 0 can sell it at time 0 at a fixed price S_0^i, but must pay at time 1 some specified amount S_1^i (contingent on the random situation in the future) to the one who has purchased the contract. A trader who has written, say, 15 standardized contracts of type i and sold them at the current price S_0^i at time 0, must pay $15 \cdot S_1^i$ at time 1.

In this example, the ith position of the portfolio at time 0 is $-15 \cdot S_0^i$, and the ith position of the portfolio at time 1 is $-15 \cdot S_1^i$. The trader who has written the contract gains

$$(-15 \cdot S_1^i) - (-15 \cdot S_0^i) = 15 \cdot (S_0^i - S_1^i)$$

if $S_0^i - S_1^i > 0$ and loses this amount if $S_0^i - S_1^i < 0$. Such transactions might be motivated by the difference of the subjective expectations of the seller and the buyer of contract i. The former expects that $S_0^i - S_1^i$ will be positive (the price goes down), while the latter hopes it will be negative (the price goes up).

1.3 Return on a Portfolio

Initial and Terminal Values of a Portfolio Suppose that, at time 0, an investor constructs a portfolio $x = (x_1, x_2, \ldots, x_N)$, i.e., invests the amount x_i in asset i. The initial wealth w_0 of the investor, or the *initial value* of the portfolio (at time 0) is equal to the sum

$$w_0 = x_1 + x_2 + \ldots + x_N.$$

Since x_i is invested in asset i, the number of units of asset i in the investor's portfolio is

$$h_i = \frac{x_i}{S_0^i}.$$

The *terminal value* (at time 1) of this portfolio is

$$w_1 = \sum_{i=1}^{N} S_1^i h_i = \sum_{i=1}^{N} \frac{S_1^i x_i}{S_0^i} = \sum_{i=1}^{N} \frac{S_1^i - S_0^i}{S_0^i} x_i + \sum_{i=1}^{N} x_i = \left(\sum_{i=1}^{N} R_i x_i \right) + w_0.$$

Thus we obtained

$$w_1 = w_0 + \sum_{i=1}^{N} R_i x_i.$$

Consequently, the difference $w_1 - w_0$ between the terminal and initial values of the portfolio (capital gain) can be expressed as follows:

$$w_1 - w_0 = \sum_{i=1}^{N} R_i x_i. \tag{1.1}$$

Normalized Portfolios An investor is usually interested in the question: in what *proportions* should an amount of money $w_0 > 0$ be distributed between the assets $i = 1, 2, \ldots, N$? To analyze this question it is sufficient to consider portfolios $x = (x_1, \ldots, x_N)$ for which

$$x_1 + \ldots + x_N = 1.$$

Such portfolios are called *normalized*.

Negative Proportions? The term "proportions" mentioned above should be used with caution. Usually, this term is associated with numbers $p_1, \ldots, p_N$ such that $p_i \geq 0$ and $\sum p_i = 1$. In the present context, we do not assume that the proportions $x_1, \ldots, x_N$ of wealth invested in assets $i = 1, \ldots, N$ (positions of a normalized portfolio) are necessarily non-negative. Portfolio positions might be long or short, and so the numbers x_i might have positive and negative signs. However, the sum of these numbers, as long as they are regarded as proportions, is always equal to one.

Return on a Normalized Portfolio The main focus of the theory of portfolio selection is on investment proportions. Hence we will basically deal with normalized portfolios. For a normalized portfolio $x = (x_1, \ldots, x_N)$ its *return* is defined as

$$\frac{w_1 - w_0}{w_0} \tag{1.2}$$

or

$$\sum_{i=1}^{N} R_i x_i . \tag{1.3}$$

These numbers *coincide*, as long as the portfolio x is normalized. Indeed, then $w_0 = 1$, and so

$$\frac{w_1 - w_0}{w_0} = w_1 - w_0 = \sum_{i=1}^{N} R_i x_i$$

[see (1.1)].

Return on a Portfolio: The General Case For a general (not necessarily normalized) portfolio, the numbers (1.2) and (1.3) might be different. We will associate the term "*return*" on a portfolio $x = (x_1, \ldots, x_N)$ with the number defined by (1.3). To emphasize the distinction between (1.2) and (1.3) in the general case, we will call (1.2) the *net return* on the portfolio x. Note that (1.2) is defined only if $w_0 \neq 0$ (one cannot divide by $w_0 = 0$), while (1.3) is defined always.

Consider the simplest possible portfolio:

$$e_j = (0, 0, \ldots 0, 1, 0, \ldots 0).$$

(Here and in what follows, e_j stands for the vector whose coordinates are equal to 0, except for the jth coordinate which is equal to 1.) This portfolio does not contain any assets except j, the holding of this asset being worth €1. This portfolio is normalized, hence its net return is equal to its return, and we have:

$$x = e_j \Rightarrow \sum_{i=1}^{N} R_i x_i = R_j .$$

Thus the return on the portfolio e_j is equal to R_j, the return on asset j.

Computing Net Return For a portfolio $x = (x_1, \ldots, x_N)$ with $w_0 = \sum_i x_i \neq 0$, the net return equals

$$\frac{w_1 - w_0}{w_0} = \frac{\sum_i R_i x_i}{\sum_i x_i} = \sum_i R_i w_0^i ,$$

where

$$w_0^i = \frac{x_i}{\sum_j x_j}$$

is the proportion of wealth (at time 0) invested in asset i.

Thus the *net return on a portfolio depends only on the proportions* of wealth invested in different assets.

Self-Financing Portfolios Portfolios $x = (x_1, \ldots, x_N)$ with zero initial value

$$x_1 + \ldots + x_N = 0$$

are called *self-financing*. How can these portfolios be created? For example, an investor can borrow some amount of money from a bank and buy some positive amounts of all the other assets. Then, if the first position x_1 of a portfolio $x = (x_1, \ldots, x_N)$ describes the investor's bank account, this position will be negative. The other positions will be positive, and the sum $\sum x_i$ will be zero. Clearly, in addition to borrowing, the investor may use the operation of short selling, which is permitted in the idealized market under consideration (but not always permitted in real markets). Note that the return

$$\sum_i x_i R_i$$

on a self-financing portfolio $(x_1, \ldots, x_N)$ is well-defined, while the net return $(w_1 - w_0)/w_0$ is not (because $w_0 = 0$). Since for a self-financing portfolio the initial value w_0 is equal to zero, we obtain

$$w_1 - w_0 = w_1, \tag{1.4}$$

and so *the return on a self-financing portfolio is equal to its terminal value* w_1.

1.4 Mathematical Notation

Notation: Scalar Product A mathematical comment is in order. According to our main definition, the return on a portfolio $x = (x_1, \ldots, x_N)$ is given by the formula

$$R_x = \sum_{i=1}^{N} x_i R_i.$$