Paul Urban

Topics in Applied Quantum-electrodynamics

Topics in Applied Quantumelectrodynamics

Paul Urban

1970
Springer-Verlag
Wien · New York

Professor Dr. PAUL URBAN
Institute of Theoretical Physics
University of Graz

With 55 Figures

Library of Congress Catalog Card Number 71-98304
Printed in Austria

Title No. 9260

Preface

These lectures represent a condensation of a number of colloquia, seminars and discussions held at the Institute of Theoretical Physics of the University of Graz during the last years and epitomize the principal lines of research undertaken by my group. From the very beginning of my appointment at the University of Graz in 1947 I have been concerned with the task of bringing up a relatively small group of scientifically interested and open-minded co-workers and of stimulating them to sound scientific research. Since 1930 I myself have dealt with subjects of the kind treated in these lectures, to which I was introduced by my late friend and teacher TH. SEXL. But also as assistant and co-worker of E. FUES and H. THIRRING I frequently worked on these problems, constantly using new methods and lines of approach. During the last years of the war and the first ones afterwards I had the fortunate opportunity to receive many stimulating ideas and comments on my work from A. SOMMERFELD on the occasion of my frequent visits to Munich. Especially this last period, although partially connected with personal difficulties and troubles of many kinds stemming from the turbulence of lost-war readjustments, I consider to be one of the most valuable times in my life. The experiences which I accumulated then I later tried to put into effect, at least to a limited extent, in order to create a productive climate for research in the spirit of A. SOMMERFELD and his school of thought. The number of my students who already hold respected positions in the scientific community at least give me the confidence that my work and my efforts were not in vain.

The following lectures are divided into two parts:

1. Electron Scattering and Nucleon Form Factors
2. Radiative Corrections

It was my intention in writing this summary not only to refer to the work done at my Institute but also to give an account of related research of many colleagues which seemed important to me. In addition, experimental results are frequently included for comparison together with a discussion of the deviations which occasionally appear. Especially the second part contains basic computations which are required for the design of experimental arrangements of current interest.

In the compilation of the text I was assisted in manifold ways by the members of my Institute. The scientific achievements of these co-workers are documented by their papers included in the reference list and give evidence of their diligence and talent. In editing these lectures especially Mr. P. PESEC and Mr. F. WIDDER were of dedicated help. I would also like to thank my friends and colleagues Prof. T. ERBER and Prof. R. ROHRLICH for a critical reading of the manuscript. The typing of manuscript was done perfectly and within shortest time by my secretary of many years, Mrs. ANNELIESE KÜHNELT; with the same skill she is doing so for our annual "Schladminger Universitätswochen". To all of them I want to express my sincerest thanks.

Graz, Fall 1969 PAUL URBAN

Contents

Part I

Electron-Scattering and Nucleon Form Factors

I. The Dirac-Foldy-Wouthuysen Transformation 1
1. The Dirac Equation 1
2. The DFW Transformation (Free Case) 6
3. The Foldy-Heisenberg (F-H) Picture 10
4. The Foldy-Interaction (F-I) Picture 14
5. Connection between DFW and Lorentz Transformation 18
6. Connection between F-H and F-I Picture 20
7. Ambiguities of the F-H Picture 21
8. Application of the DFW Transformation 22
Literature 24

II. Determination of Proton-Form Factors Derived from Electron-Proton-Scattering 25
1. Introductory Remarks 25
2. Relativistic Electron-Proton-Scattering (Rosenbluth Formula) 27
3. Electromagnetic Form Factors of the Proton 38
Literature 44

III. Determination of the Neutron Form Factors Derived from Quasielastic Electron-Deuteron-Scattering 45
1. Introduction 45
2. Quasielastic Electron-Deuteron-Scattering 48
3. Calculation of the Cross-Section in the Lab.-System 52
4. Influence of the Deuteron Model 67
5. Final State Corrections 71
Appendix 77
Literature 79

IV. Calculation of Nucleon Form Factors in Dispersion Theory 81
Literature 95

Part II

Radiative Corrections

Introduction 96

I. Classical Radiation of Long-Wavelength Photons 101
1. Introduction 101
2. Emission of Soft Photons 106
3. Experimental Cross-Section and Radiative Corrections 116
4. Calculations of Radiative Corrections 126

II. Summation over Soft-Photon Contributions 145
1. Separation of Soft and Hard Photons 145
2. Virtual Soft Photons 152
3. Real Soft Photons 156
4. Summation over Soft Photons 160

III. Radiative Corrections in the Framework of Quantumelectrodynamics 168
1. Infrared Divergences 169
2. The Canceling of Infrared Divergences 176
3. Summation of the "Infrared" Contributions 179

IV. Examples 186
1. Electron-Proton-Scattering 186
2. Inelastic Electron-Scattering 216
Literature 226

Appendix

A. The Green-Functions of the Klein-Gordon Equation and the Dirac Equation 228

B. Theory of Bosons, Klein-Gordon Equation 234
1. Derivation of the Klein-Gordon Equation 234
2. Scalar Wave Functions 235
3. Wave Functions for Particles with Spin 1 240

C. Theory of Fermions, Dirac Equation 247
1. Relativistic Wave-Equation for Fermions 247
2. Charge Conjugation 250
3. Solutions for a Free Particle 253

Part I

ELECTRON-SCATTERING AND NUCLEON FORM FACTORS

I. The Dirac-Foldy-Wouthuysen Transformation+

1. The Dirac Equation

The Dirac-Foldy-Wouthuysen (DFW) transformation will be discussed in its application to the Dirac equation, where it is best known; essentially the same conclusions, however, also hold in case of the Klein-Gordon equation [2], [3].

The relativistic motion of spin $\frac{1}{2}$ - particles in an electromagnetic potential $A_\mu(x)$ is governed by the Dirac equation; with the notation ($\hbar = c = 1$)

$$(ab) = a^\mu b_\mu = a_\mu b^\mu = a^\mu b^\nu g_{\mu\nu} = a^o b^o - \vec{a} \cdot \vec{b} ;$$

$$\gamma^\mu = (\gamma^o, \vec{\gamma}) ;$$

$$\not{a} = a^\mu \gamma_\mu ;$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} ;$$

$$\gamma^o = \beta = \begin{pmatrix} I & o \\ o & -I \end{pmatrix} , \qquad \vec{\gamma} = \beta \vec{\alpha} , \qquad \vec{\alpha} = \begin{pmatrix} o & \vec{\sigma} \\ \vec{\sigma} & o \end{pmatrix} ; \qquad (1,1)$$

+ This transformation has been discussed by P.A.M. Dirac [1] as early as in 1934 (private communication by Prof. P.E. Wigner).

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

it has the form $[A_\mu = (\Phi, \vec{A})]$:

$$(i\not\partial - e\not A(x) - m)\,\psi(x) = 0\,. \tag{1,2}$$

Here $\psi(x)$ is a four-component spinor, γ^μ we can take as in the definition (1,1) or according to an equivalent representation; in any case they fulfil the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\,. \tag{1,3}$$

By the iteration of (1,2) we obtain a wave equation which differs from the Klein-Gordon equation by the coupling between spin and electromagnetic field:

$$(i\not\partial - e\not A(x) + m)(i\not\partial - e\not A(x) - m)\,\psi(x) =$$
$$= [(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - m^2 + \frac{e}{2} F^{\mu\nu}\sigma_{\mu\nu}]\,\psi(x) = 0\,, \tag{1,4}$$

where we used

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \quad F^{ok} = E^k, \quad F^{kl} = \epsilon_{klm} B^m \;;$$
$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\,.$$

The term $\frac{e}{2} F^{\mu\nu}\sigma_{\mu\nu}$ in (1,4) corresponds to the term $\frac{e}{m}\vec{\sigma}\cdot\vec{B}$ of the nonrelativistic Pauli equation. In momentum space, through the transformation

$$\psi(x) = \frac{1}{(2\pi)^2}\int d^4p\, e^{-ipx}\,\psi(p)\,, \tag{1,5}$$

equation (1,2) has the form

$$(\not p - e\not A - m)\psi(p) = 0\,, \tag{1,6}$$

which for free particles reads

$$(\not p - m)\,\psi(p) = 0\,. \tag{1,7}$$

Equation (1,7) is a short-hand notation for actually four homogeneous linear equations involving the four components of the spinor $\psi(p)$; the determinant of coefficients is

$$\det(\not p - m) = (p^2 - m^2)^2 = (p_o - E)^2 (p_o + E)^2, \qquad (1,8)$$

where $E = +\sqrt{\vec{p}^{\,2} + m^2}$. Equation (1,7) therefore possesses two solutions to each value of p_o ($\pm E$), corresponding to the spin orientations.

As usual the solutions corresponding to negative energy ($p_o = -E$) are interpreted as antiparticles in the framework of Dirac's hole-theory which establishes a complete symmetry between particles and antiparticles (charge-conjugation symmetry).

Exactly this existence of antiparticles, which is necessarily connected with each local relativistic covariant wave equation (this is part of the conclusions drawn from the PCT theorem), leads to difficulties in the interpretation and application of the Dirac equation.

Writing (1,7) in Hamiltonian form

$$P_o \psi = H \psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\, \psi, \qquad (1,9)$$

(in the following p_μ may stand for either p_μ or $i\partial_\mu$) we can for example investigate the operator $\dot{\vec{x}}$ which may be interpreted as velocity operator:

$$\vec{v} = \dot{\vec{x}} = i[H, \vec{x}] = \vec{\alpha}, \qquad (1,10)$$

by virtue of

$$[x_k, p_l] = i\delta_{kl}. \qquad (1,11)$$

Since $\alpha_i^2 = 1$ the magnitude of the velocity $\vec{v}$ is always equal to the velocity of light c. In addition, the different components of the velocity cannot be defined simultaneously since $[\alpha_i, \alpha_j] \neq 0$. This, however, contradicts the possibility of observation of the velocity. Further problems arise in applications if one wants to employ nonrelativistic wave functions, e.g. from nuclear physics, together with the Dirac equation.

Thus it is evident that another representation for the Dirac equation has to be found in order to make possible a physical interpretation. A Dirac-particle with positive energy has to be represented by only two vectors in Hilbert space corresponding to its two possible spin orientations. Therefore two components of the four-component wave function in Dirac's theory are superfluous, and we have to find a transformation reducing the Dirac equation to a two-component equation, for example the nonrelativistic Pauli-theory.

Large and Small Components, Pauli Equation

The Dirac equation (1,6) can be rewritten as two coupled equations by expressing the wave function in terms of two-component spinors φ and χ:

$$\psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix} ; \tag{1,12}$$

$$(\vec{\sigma} \cdot (\vec{p} - e\vec{A}))\, \chi + (e\Phi + m)\, \varphi = E\varphi,$$

$$(\vec{\sigma} \cdot (\vec{p} - e\vec{A}))\, \varphi + (e\Phi - m)\, \chi = E\chi . \tag{1,13}$$

Solving the last equation for χ we get

$$\chi = \frac{1}{2[m + \frac{1}{2}(E - e\Phi - m)]}\, \vec{\sigma} \cdot (\vec{p} - e\vec{A})\, \varphi , \tag{1,14}$$

and inserting this in (1,13) we arrive at an equation exactly equivalent to the Dirac equation:

$$[\vec{\sigma}(\vec{p} - e\vec{A})\, \frac{1}{2[m + \frac{1}{2}(E - e\Phi - m)]}\, \vec{\sigma} \cdot (\vec{p} - e\vec{A}) + e\Phi]\, \varphi = (E - m)\, \varphi . \tag{1,15}$$

In the nonrelativistic limit we have

$$E - m,\ e\Phi,\ \vec{p},\ e\vec{A} \ll m\,;\ m + \tfrac{1}{2}(E - e\Phi - m) \approx m\,,$$

and therefore also

$$\chi \ll \varphi\,,$$

or, in other words, the ratio of χ and φ is of order $\frac{p}{m}$ or $\frac{v}{c}$. Hence χ and φ are called small and large components respectively. The terms "even" and "odd" operator are closely connected with this:

An operator is called "even" if it contains no matrix elements connecting large and small components, as e.g. $\vec{\sigma}$, β. An even operator commutes with β.

An "odd" operator, however, contains nonvanishing matrix elements connecting large and small components; it anticommutes with β.

We now discuss the nonrelativistic limit of (1,15), thereby neglecting the small components. In this approximation to order v/c we get the eigenvalue equation

$$H_{NR}\,\varphi = E_{NR}\,\varphi\,, \tag{1,16}$$

with the Hamiltonian of the Pauli-theory

$$H_{NR} = \frac{1}{2m}\,\vec{\sigma}\cdot(\vec{p}-e\vec{A})\,\vec{\sigma}\cdot(\vec{p}-e\vec{A}) + e\,\Phi =$$

$$= \frac{1}{2m}\,(\vec{p}-e\vec{A})^2 - \frac{e}{2m}\,(\vec{\sigma}\cdot\vec{B}) + e\,\Phi\,. \tag{1,17}$$

It is of course possible to evaluate higher orders in v/c by this method; no eigenvalue equation, however, results and the Hamiltonian ceases to be Hermitian. Foldy and Wouthuysen therefore suggested another method by which Dirac's theory can be approximated to any order in v/c by means of a two-component theory [4].

2. The DFW Transformation (Free Case)

Foldy and Wouthuysen [5] found a unitary transformation for the diagonalization of the Hamilton-operator. Then the Dirac equation decouples into two-component equations, one for positive and one for negative energy. For free particles large and small components are completely decoupled to any order in v/c; the transformation can be given in closed form.

This unitary transformation has the form+

$$\varphi(p) = U(p)\,\psi(p), \tag{1,18}$$

with

$$U(p) = \exp\{\frac{\beta}{2} \text{ arc tan } (\frac{\vec{\alpha}\cdot\vec{p}}{m})\} = \frac{E+m+(\vec{\gamma}\cdot\vec{p})}{\sqrt{2E(E+m)}} \,. \tag{1,19}$$

From (1,19) it can be seen immediately that

$$U^{+}(p) = U(-p) = U^{-1}(p)\,. \tag{1,20}$$

The Dirac equation (1,9)

$$P_0 \psi = H_0 \psi = (\vec{\alpha}\cdot\vec{p} + m\beta)\,\psi,$$

then is transformed into

$$P_0 \varphi = H'_0 \varphi, \tag{1,21}$$

where

$$H'_0 = U(p)\,H_0 U^{+}(p) = \beta\sqrt{p^2 + m^2}\,. \tag{1,22}$$

Since this transformed Hamiltonian commutes with β,

+ The last expression is found with an expansion of arc tg $\frac{\vec{a}\vec{p}}{m}$ and of exp{/} with the result that $U(p) = \cos(\frac{1}{2} \text{ arc tg } \frac{p}{m}) + \frac{\vec{\gamma}\vec{p}}{p} \sin(\frac{1}{2} \text{ arc tg } \frac{p}{m})$ which gives with tg $\omega = \frac{p}{m}$ and cos $\omega = \frac{m}{E}$ the final form.

$[H'_0, \beta] = 0$, then $\frac{1}{2}(1 \pm \beta)$ are projection operators for states of positive or negative energy respectively, where two components vanish identically in each case. Therefore the resulting equations involve two components only:

$$\varphi = \binom{\varphi_+}{\varphi_-}, \quad p_0 \varphi_\pm = \pm E\, \varphi_\pm . \tag{1,23}$$

The relevant equations decompose further still since also σ_3 commutes with H'_0, and thus $\frac{1}{2}(1 \pm \sigma_3)$ again represents a projection operator.

In order to get a better understanding of the transformation obtained it is convenient to discuss the new wave function in configuration space, as has been done by Foldy and Wouthuysen. We can decompose any wave function as

$$\psi(\vec{x}) = \int u(\vec{p}')\, e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{p}' = \psi_+(\vec{x}) + \psi_-(\vec{x});$$

$$\psi_+(\vec{x}) = \int \tfrac{1}{2}\left(1 + \frac{\beta m + \vec{\alpha}\cdot\vec{p}'}{E_{p'}}\right) u(\vec{p}')\, e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{p}', \tag{1,24}$$

$$\psi_-(\vec{x}) = \int \tfrac{1}{2}\left(1 - \frac{\beta m + \vec{\alpha}\cdot\vec{p}'}{E_{p'}}\right) u(\vec{p}')\, e^{i\vec{p}'\vec{x}}\, d^3\vec{p}' .$$

Here ψ_+ represents the wave function for positive, ψ_- the one for negative energy. We now perform the DFW transformation:

$$\varphi_+(\vec{x}) = U\psi_+(\vec{x}) =$$

$$= \left(\frac{1+\beta}{2}\right) \int \sqrt{\frac{E_{p'}}{2(E_{p'}+m)}}\left[1 + \frac{\beta m + \vec{\alpha}\cdot\vec{p}'}{E_{p'}}\right] u(\vec{p}')\, e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{p}' ;$$

$$\varphi_-(\vec{x}) = U\psi_-(\vec{x}) =$$

$$= \left(\frac{1-\beta}{2}\right) \int \sqrt{\frac{E_{p'}}{2(E_{p'}+m)}}\left[1 - \frac{\beta m + \vec{\alpha}\cdot\vec{p}'}{E_{p'}}\right] u(\vec{p}')\, e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{p}' . \tag{1,25}$$

From this we clearly see that the upper components in (1,25) correspond to positive, the lower ones to negative energy.

Inserting the inverse Fourier transform

$$u(\vec{p}\,') = \frac{1}{(2\pi)^3} \int \psi(x')\, e^{-i\vec{p}\,'\cdot\vec{x}\,'}\, d^3\vec{x}\,' ,$$

we get the relation between old and transformed wave function

$$\varphi(\vec{x}) = \int K(\vec{x},\vec{x}\,')\; \psi(\vec{x}\,')\, d^3\vec{x}\,' = \varphi_+(\vec{x}) + \varphi_-(\vec{x}) \tag{1,26}$$

where

$$K(\vec{x},\vec{x}\,') = \frac{1}{(2\pi)^3} \int \sqrt{\frac{E_{p'}}{2\,(E_{p'}+m)}}\; [1 +$$

$$+ \frac{m + (\vec{\gamma}\cdot\vec{p}\,')}{E_{p'}}]\; \exp\{i\vec{p}\,'\cdot(\vec{x}-\vec{x}\,')\}\, d^3\vec{p}\,' . \tag{1,27}$$

Because of the momentum dependence of the DFW transformation the kernel $K(\vec{x},\vec{x}\,')$ exhibits in its spatial dependence no δ-function; the transformation is not a point-transformation. Had the wave-function in the old representation been localized in a point the transformed wave-function would be smeared out over a finite range (of the Compton wavelength's order of magnitude) [6].

An interesting problem is the interpretation of physical quantities in this new representation. In case of the position-operator we ask for the operator $\vec{X}$ of the old representation which corresponds in the DFW representation to the usual position operators $\vec{x}$, and find with $[x_i, f(p)] = i\frac{\partial}{\partial p_i} f(p)$

$$\vec{X} = U^+\, \vec{x}\, U = \vec{x} + \frac{i\beta\vec{\alpha}}{2E} - \frac{i\beta(\vec{\alpha}\cdot\vec{p})\vec{p} + (\vec{\sigma}\times\vec{p})E}{2E^2(E+m)} . \tag{1,28}$$

The time derivative in the new representation is

$$\frac{d}{dt}\vec{x} = i\,[H'_o, \vec{x}] = -i\,[\vec{x}, \beta E] = \beta\frac{\vec{p}}{E}, \qquad (1,29)$$

and in the old representation

$$\frac{d\vec{X}}{dt} = i\,[H, \vec{X}] = U^{+}\frac{d}{dt}\vec{x}\,U = \frac{\vec{p}}{E}\,\frac{m\beta + \vec{\alpha}\vec{p}}{E}. \qquad (1,30)$$

The operator $\frac{1}{E}(m\beta + \vec{\alpha}\vec{p})$ - applied to a positive or negative energy wave function - has the value $+1$ or -1 respectively. So we get the result that the velocity operator for positive energy states is $+\frac{p}{E}$ in either representation.

The operator $\vec{x}$ has been interpreted by Newton and Wigner [7] as the position operator appropriate for describing localized states. It has the properties

$$[X_i, X_j] = 0 \;\; ; [X_i, p_j] = i\,\delta_{ij} \qquad (1, 31)$$

which can be verified in a way similar to (1,30).

3. The Foldy-Heisenberg (F-H) Picture

In addition Foldy and Wouthuysen have developed a method whic enables a step-by-step diagonalization of the Hamiltonian in any arbitrary order of v/c or 1/m. This method is important when interactions are present.

The Dirac-Hamiltonian can be decomposed into an even and odd part:

$$H = m\beta + \vec{\alpha}\vec{p} = m\beta + \epsilon + o \; ; \tag{1,32}$$

$$\{\beta, o\} = 0, \; [\beta, \epsilon] = 0 \; .$$

These odd and even parts we can write

$$o = \frac{\beta}{2}[\beta, H] = \tfrac{1}{2}(H - \beta H \beta),$$

$$\epsilon = \frac{\beta}{2}(\{\beta, H\} - 2m) = \tfrac{1}{2}(H + \beta H \beta) - \beta m \; . \tag{1,33}$$

We now perform a unitary transformation with

$$U = e^{iS} \; ,$$

where

$$S = -\frac{i\beta o}{2m} = -\frac{i}{4m}(\beta H - H\beta) \; , \tag{1,34}$$

so that

$$\psi' = U\psi, \; H' = U\left(H - i\frac{\partial}{\partial t}\right)U^{+} \; .$$

The new Hamiltonian H' then has the form

$$H' = e^{iS}\left(H - i\frac{\partial}{\partial t}\right)e^{-iS} =$$

$$= H + i[S, H] - \tfrac{1}{2}[S, [S, H]] - \frac{\partial S}{\partial t} - \frac{i}{2}\left[S, \frac{\partial S}{\partial t}\right] \ldots . \tag{1,35}$$

where we took into account a possible explicit dependence of S on time. Evaluation of the terms leads to