Fundamentals of Mechanical Vibrations

Liang-Wu Cai

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FUNDAMENTALS OF MECHANICAL VIBRATIONS

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Series Preface

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Preface

Why a new book on mechanical vibrations?

Mechanical vibration is a core subject in mechanical engineering that has been taught for many decades, with many old classics and many more excellent contemporary textbooks available in a rather crowded market. However, after teaching this subject for several years, I feel that there are two major challenges facing the current generation of engineering students studying vibrations, and, till this day, I am still unable to identify a textbook that satisfactorily addresses these challenges. This book represents my efforts in addressing these challenges.

A vibration analysis of a system starts from the equation(s) of motion for the system. The first major challenge facing the students is reaching this starting point. Although a prerequisite, the undergraduate Newtonian dynamics course does not adequately prepare students to derive the equation(s) of motion for a system that is slightly more complex than a mass–spring–dashpot system. Without teaching them on how to get from "here" (a given system) to "there" (the equations of motion), most textbooks parade through the analyses starting from "there."

In my view, Lagrangian dynamics is the ideal approach to reach this starting point. This view can be corroborated by the fact that almost all vibration textbooks contain a brief section, often in less than 10 pages, on Lagrangian dynamics, often just before vibration analyses of multi-degree-of-freedom systems. Lagrangian dynamics is typically a graduate-level topic that is rather mathematical and abstract for undergraduate students. The abstract nature compounded with the brevity in coverage does not prepare the students to reach the starting point with confidence.

I am partial to Lagrangian dynamics in part because of where I came from. When I was a doctoral student at Massachusetts Institute of Technology, for many years, I was a teaching assistant for the dynamics course in which Lagrangian dynamics was taught to sophomores, whose previous exposure to dynamics was college physics. Professor James H. Williams, Jr., who taught the course at that time, was also writing his own textbook on Lagrangian dynamics (*Fundamentals of Applied Dynamics*, John Wiley & Sons, 1996), which was the culmination of his years of award-winning teaching of Lagrangian dynamics to undergraduate students from a classic textbook (*Dynamics of Mechanical and Electromechanical Systems*, by Crandall, Karnopp, Kurtz, and Pridmore-Brown, McGraw-Hill, 1968). I was also helping him on the preparation for the solutions manual. I observed how such a difficult topic was taught to sophomores without losing its rigor. That experience gave me the belief that Lagrangian dynamics can be taught at undergraduate level, and Professor Williams had presented a successful paradigm in his textbook.

Another major challenge facing students is the symbolic analysis. In a vibration analysis, the solution is almost always in the form of an analytical expression. The current generation of engineering students, with sophisticated scientific calculators at their handy disposal, have grown comfortable with numbers: given a problem with a set of numbers, they apply a solution process to yield another number called the *answer*. The answer, right or wrong, represents a closure, as well as a sense of accomplishment. But in a symbolic analysis, the lack of numerical values given for the parameters in the problem appears to pull them away from the physical sense of the system, and the lack of a numerical answer makes them feel uncertain whether they have reached the final solution to the problem.

In my view, a graphical representation of an analytical solution could fill in as "the answer" to a symbolic problem. Graphing an analytical expression involves mundane details and some important steps such as nondimensionalizing the expression. The mundane details can be eliminated by incorporating mathematical software such as MatLab into the learning process, essentially giving the graphing capability in their handy disposal. This could reduce their uneasiness toward symbolic analysis and eventually learn this vitally important skill.

I feel that the introductory course on vibration is a perfect place to start training students for this skill. On the one hand, by junior or senior years, students have acquired a sense for the importance of the symbolic analysis: they have the motivation. On the other hand, many traditional vibration analysis topics can be enlivened by the computational capabilities brought by MatLab. Topics such as steady-state responses due to general periodic loadings by using Fourier expansions and the convolution integral as a general-purpose numerical method for an arbitrary transient loading can benefit from numerical computations.

Empowering students with these skills and capabilities does not require the commitment of a significant amount of time and effort, but requires the right instructional materials. After teaching the vibration course at Kansas State University for a couple of years using different textbooks, I started writing my own lecture notes.

Starting from a clean slate, I focused on the following set of learning objectives that I deem the most essential for prospective engineers:

- Being able to reach the starting point of a vibration analysis with confidence.
- Being able to establish a simple mathematical model for vibration analysis of real-world structures.
- Being able to handle the mathematics of vibration analyses, and, with the aid of a computer if necessary, to interpret the physical meanings of the results.
- Being able to tackle mathematical analysis symbolically.
- Being ready to use more powerful simulation tools when the needs call for.

The materials presented in this book are accumulated and refined over the past 10-plus years. To help the students achieving the above set of learning objectives, this textbook incorporates the following carefully designed pedagogical features:

- Staying true to the fundamentals. This book emphasizes providing thorough and clear explanations to the fundamental concepts and theories. It does not stray into advanced topics that are more suitable for advanced studies in graduate school. The time (and space) saved can be better spent on honing in the skills of symbolic analysis and the capabilities afforded by MatLab.

- Treating Lagrangian dynamics with rigor while maintaining accessibility. By limiting the consideration to Newtonian particles and rigid bodies, Lagrangian dynamics is tuned down its generality and the associated mathematical complexity. Students having the prerequisite of Newtonian dynamics are offered a fresh perspective. One of the recurring comments I received from students is that they "finally learned the dynamics!"
- Provoking higher level learning and thinking by enforcing symbolic analysis. MatLab is used initially as merely a tool for visualizing analytical solutions. By limiting the use of MatLab to this rudimentary and humble goal, students are greatly empowered in their analytical skills.
- Instilling the engineers' philosophy that having a solution to a problem is not the end to the problem; in contrary, it is the beginning of an exploration. In every vibration example, a section entitled *Exploring the Solution with MatLab* is dedicated to curiosity-driven observations and explorations.
- Expanding the horizon by exploiting the capabilities of MatLab. MatLab is gradually used as a computational tool for analyzing more complicated problems such as modeling a fascinating phenomenon of levitating Slinky. In the end, a complete finite element analysis code is developed in MatLab for vibration analysis of one-dimensional beams.
- Providing a consistent approach to modeling engineering systems. This book offers the following components to prepare the students for the industry:
  - A systematic procedure for establishing lumped-parameter models for simple engineering structures and systems. This gives students the skill to produce a quick and reasonably accurate estimate of practical problems.
  - The same lump-parameter modeling procedure is used for finite element formulation. This serves two purposes: it gives students an understanding of the working principle of the finite element method; and it boosts students the confidence on the effective lump-parameter modeling.
  - A set of illustrated tutorials for vibration analyses of a real structure using a commercial finite element analysis software package is also provided.

It is my sincere hope that this new textbook will bring a new breeze to the market that is crowded with many excellent textbooks over the past few decades. I wholeheartedly welcome all criticisms and suggestions to make it a better textbook.

In writing this book, first and foremost, I am greatly indebted to my former advisor at MIT, Professor James H. Williams, Jr., for offering me the opportunity as the teaching assistant for his class and as an assistant for preparing his textbook. His teaching of dynamics forms the basis of my firm belief that Lagrangian dynamics should be the starting point of the book, which inevitability bears the signature of his paradigm. Over the years, various draft versions of this book have been used in my class, and many students have provided valuable feedback. In particular, Ms. Congrui Jin and Mr. Ryan Cater offered extensive corrections to early versions. I would also like to thank Mr. Paul Petralia, senior acquisitions editor at Wiley, for his patience and encouragements, and Clive Lawson (UK), Preethi Belkese (India), and other editors at Wiley for their assistance. Finally, I would like to thank my wife, Huimin, for her unconditional support and love during this endeavor.

Liang-Wu Cai
September 2015

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A Crash Course on Lagrangian Dynamics

1.1 Objectives

This chapter presents the fundamental concepts in Lagrangian dynamics and outlines a procedure for deriving the equation(s) of motion for holonomic mechanical systems using Lagrange's equation. Extensive examples are presented to cover a large variety of mechanical systems containing particles and rigid bodies. Finally, the procedures for finding the equilibrium position(s) and linearizing the equation(s) of motion in preparation for vibration analysis are also presented.

1.2 Concept of "Equation of Motion"

An *equation of motion* for a mechanical system is a differential equation that governs the changes in positions of components in the system with respect to time.

There are three key phrases in the above definition. Phrases *differential equation* and *with respect to time* specify a particular mathematical form for the equation and distinguish the equation of motion from its solution. That is, an equation of motion is a differential equation involving time derivatives. The phrase *positions* specifies a particular kinematic quantity, the most fundamental one from which other measures of motion, namely, displacements, velocities, and accelerations, can be obtained.

The concept of the *equation of motion* has suffered a degree of misuse in some dynamics textbooks. In these textbooks, Newton's second law, which many of us conveniently recite as $\boldsymbol{F} = m\boldsymbol{a}$, is called the equation of motion. In fact, Newton's second law is the fundamental physical law that governs the motion of any physical system. It is often a crucial tool to use in obtaining the equation(s) of motion for a system. But it is too primitive a form to be called an equation of motion. In our daily lives, we do not call a foundation as a structure. The same goes here.

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In the mean time, we must keep our minds open to the notion that there are other ways to establishing the equation of motion for a system. *Lagrangian dynamics* is one such alternative.

Before jumping into the details of Lagrangian dynamics, let us look at the way the equation of motion for a system can be obtained by using Newton's second law.

■ Example 1.1: Simple Mass–Spring–Dashpot System

Derive the equation of motion for the mass–spring–dashpot system as shown in Fig. 1.1.

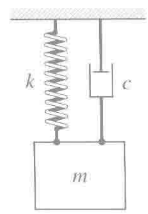

Figure 1.1 Mass–spring–dashpot system

□ Solution 1:

- Define x as the downward displacement of the mass, measured from its position when the spring is unstretched.
- *Kinematics of Mass m*: This example is simple enough to allow us to write directly:

$$\text{velocity} = \dot{x} \qquad \text{and} \qquad \text{acceleration} = \ddot{x} \tag{a}$$

  where, as in dynamics, an overhead dot represents the time derivative, and overhead double-dot represents the second derivative with respect to time t.
- *Kinetics of Mass m*: A free-body diagram can be drawn, as in Fig. 1.2, to show all the forces acting on the mass in a generic instant in time when the system is in motion. In Fig. 1.2, F_s and F_d are the forces exerted by the spring and the dashpot, respectively. Applying Newton's second law based on the free-body diagram in Fig. 1.2 gives

$$mg - F_s - F_d = m\ddot{x} \tag{b}$$

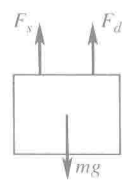

Figure 1.2 Free-body diagram for mass m at a generic time instant

- *Constitutive Relations*: Besides the mass, the system contains a spring and a dashpot, whose forces are proportional to the deflection and the velocity, respectively. That is,

$$F_s = kx \qquad \text{and} \qquad F_d = c\dot{x} \tag{c}$$

- Substituting eqns. (c) into eqn. (b) gives, with a slight rearrangement,

$$m\ddot{x} + c\dot{x} + kx = mg \tag{d}$$

which is the equation of motion for this system.

□ **Solution 2:**

- Define y as the downward displacement of the mass, measured from its static equilibrium position.
- *Kinematics of Mass m*: Kinematics of mass m is unchanged, except replacing x by y.
- *Kinetic of Mass m*: The free-body diagram remains the same as in Fig. 1.2; and hence Newton's second law gives

$$mg - F_s - F_d = m\ddot{y} \tag{e}$$

- *Constitutive Relations* for system components are

$$F_s = k(y + \Delta) \qquad \text{and} \qquad F_d = c\dot{y} \tag{f}$$

where we note that the spring is already stretched at equilibrium, denoted as Δ, and that the spring force is proportional to the total amount of deformation in the spring.
- Substituting eqn. (f) into eqn. (e) gives

$$mg - ky - k\Delta - c\dot{y} = m\ddot{y} \tag{g}$$

- To eliminate Δ from eqn. (g), we look at the static equilibrium: the dashpot does not exert any force to the mass. The free-body diagram for the mass in its static equilibrium state is shown in Fig. 1.3. The equilibrium requires

$$k\Delta = mg \tag{h}$$

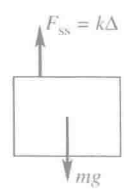

Figure 1.3 Free-body diagram for mass m at the static equilibrium

- Combining eqns. (g) and (h) gives, after a slight rearrangement,

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{i}$$

which is the equation of motion for the system.

———— · ————

This example illustrates a typical process for deriving the equation(s) of motion for a mechanical system by using Newton's second law. Through this example, we can make the following observations:

- Before we can proceed to deriving the equation, some variables (such as x and y) must be defined so that the locations of the system's components can be described. In the end, the equations of motion for the system are differential equations of these variables. Such variables are called the *generalized coordinates*. We shall study this concept more thoroughly and rigorously in the next section.
- Different definitions of the generalized coordinates result in different equations of motion for the same system. That is, the equations of motion are not unique, depending on the choice of the generalized coordinates by which the system is described.
- The following three pieces of information are generally needed: (1) kinematics of the system components at a generic time instant during the motion; (2) the properties of system components, which are called the *constitutive relations*; and (3) Newton's second law that threads all these pieces of information together.

However, there are some shortcomings in this approach of deriving the equations of motion for a system. For example, how do we know how many equations should be obtained? For a complex system, which part or parts of the system should be isolated to draw the free-body diagrams? Finding the answer to these questions is *ad hoc*: we have to look into individual systems case by case. This means that we may be able to solve one problem effortlessly, but we might stumble on the next. We all have experienced the situations in which drawing a free-body diagram reveals more unknowns than desired and calls for drawing more free-body diagrams and writing out more equations. We may also recall that, in kinematics, finding the acceleration for a particle or a point in a rigid body involves substantially more work than finding the velocity.

Lagrangian dynamics avoids most of these issues and provides a structured way to analyze a system and to subsequently obtain its equation(s) of motion. It works in a way similar to the energy method: only positions and velocities are required in the formulation. The procedure is unchanged regardless of the system's complexity. Furthermore, it can be extended to handle systems in other physical domains, such as electrical, electromagnetic, and electromechanical systems.

Deriving the equation of motion is the first step toward understanding a system. Having the equation(s) of motion in hand, the first and foremost, we can find and subsequently analyze the solution to the equation(s) of motion. This plainly stated activity, in fact, encompasses almost all subjects of study in mechanical engineering, including vibration analyses. As we shall see, vibration analyses are to find, analyze, and study the solutions to a particular class of equations of motion.

Having the equation of motion, before putting all the efforts into finding the solution, a few less ambitious things having great engineering interests can be done:

- Determine the system's equilibrium configuration(s). In the equilibrium state, the system does not move at all. For the above example, setting the velocity and acceleration to zero, the equation of motion derived in *Solution 1* would become

$$kx_{\mathrm{eq}} = mg \qquad \text{or} \qquad x_{\mathrm{eq}} = \frac{mg}{k}$$

  where x_{eq} denotes the position of the mass at equilibrium. The equation of motion derived in *Solution 2* would give $y_{\mathrm{eq}} = 0$, as expected.
- Analyze the dynamic stability and other associated behaviors of the system.

- Introduce approximations to simplify the equation(s) of motion or to obtain approximate solutions.

We discuss these topics separately later in this chapter. In fact, these things must be explored before we can proceed with vibration analyses of a system.

1.3 Generalized Coordinates

Coordinates usually are associated with a particular coordinate system and usually appear in a set. For instance, in a Cartesian coordinate system, the coordinates for a point are (x, y, z). In a polar coordinate system, the coordinates are (r, θ). Adding the word *generalized* frees us from abiding to any particular coordinate system so that we can choose whatever parameter that is convenient to describe the position of a point in a system. Hence,

A *generalized coordinate* is a parameter that is used to locate a part of a system. A generalized coordinate is a scalar quantity.

A group of parameters that is used to locate a system is a *set* of generalized coordinates. To denote a set of generalized coordinates, we follow the mathematical notation for explicitly defining a set by listing all elements contained in the set. We write, for example, $\{x_1, x_2\}$.

A *complete set* of generalized coordinates is a set of generalized coordinates that completely locates all parts of a system in all geometrically admissible configurations. A *geometrically admissible configuration* is a configuration that is allowed by the geometrical constraints in the system.

An *independent set* of generalized coordinates is a set of generalized coordinates in which if all but one of them are fixed, there still exists a continuous range of values for that unfixed generalized coordinate.

A *complete and independent set* of generalized coordinates is, literally, a set of generalized coordinates that is both complete and independent.

Because generalized coordinates can be chosen or defined almost at will, generalized coordinates for a system are not unique. For this same reason, it is utterly important to define them clearly, aided by a schematic depiction if necessary. When defining a generalized coordinate, usually the following four vital aspects should be clearly indicated for each generalized coordinate: the exact physical meaning, where it is measured from, and relative to whom, and which direction is positive.

■ Example 1.2: Particle Moves Freely in Three-Dimensional Space

A particle can move freely in a three-dimensional space. Define a set of generalized coordinates and subsequently judge whether it is a complete and independent set of generalized coordinates.

□ Solution:

Before we start, as a preparatory setup, we define a Cartesian coordinate system to facilitate the discussions that follow. The Cartesian coordinate system is fixed in space.

We define $\{x, y, z\}$ as a set of generalized coordinates, where x, y, and z are the Cartesian coordinates of the particle. Note that the positive directions for the coordinates have already been defined in the Cartesian coordinate system.

- *Is It a Complete Set?* YES, because there is only one particle in the three-dimensional space, the three coordinates completely specified a point in the three-dimensional space.
- *Is It an Independent Set?* To answer this question, we need to conduct a series of tests: if all but one of the generalized coordinates are fixed, will there be a continuous range of values for the unfixed one to change. In this problem, if x and y are fixed, z can still be freely changed along a line parallel to the z-axis. Similar conclusion can be drawn for leaving any other coordinate unfixed. So, the answer to this question is also YES.

Therefore, we conclude that $\{x, y, z\}$ as defined is a complete and independent set of generalized coordinates.

□ Discussion: Using a Coordinate System

When using coordinates in a well-established coordinate system, such as the Cartesian coordinate system in this example, as the generalized coordinates, in general, the coordinate system has already included the specifications for the directions of positive coordinates. An important task in defining a coordinate system is to specify how its origin moves, such as relative to whom, and how the coordinate axes are oriented.

■ Example 1.3: Simple Planar Pendulum

A simple pendulum is made up by a particle hung to a pivot point through a massless string. A planar pendulum is a simple pendulum whose motion is restricted to within a plane, typically the plane of paper, as shown in Fig. 1.4. Assume that the string of length l remains taut at all times. Define a set of generalized coordinates for this pendulum and subsequently judge whether it is a complete and independent set of generalized coordinates.

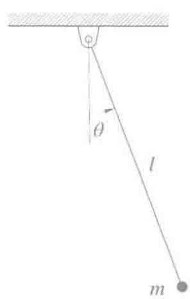

Figure 1.4 Simple planar pendulum

□ Solution 1: Angular Displacement

We define $\{\theta\}$ as a set of generalized coordinates, where θ is the angular displacement of the pendulum, measured in the counterclockwise direction from the vertical, as shown in Fig. 1.4. Note that, in the sketch, the arrow for θ goes only in one direction, indicating its positive direction.

- *Is It a Complete Set?* YES, because there is only one particle in the system and it can only move around a circle of radius l centered at the pivot point. Once the angle θ is determined, the location of the particle is uniquely determined.