

Analysis on Polish Spaces

and an Introduction to Optimal Transportation

D. J. H. GARLING

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A large part of mathematical analysis, both pure and applied, takes place on Polish spaces: topological spaces whose topology can be given by a complete metric. This analysis is not only simpler than in the general case, but, more crucially, contains many important special results.

This book provides a detailed account of analysis and measure theory on Polish spaces, including results about spaces of probability measures. Containing more than 200 elementary exercises, it will be a useful resource for advanced mathematical students and also for researchers in mathematical analysis.

The book also includes a straightforward and gentle introduction to the theory of optimal transportation, illustrating just how many of the results established earlier in the book play an essential role in the theory.

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Contents

Introduction	1
PART ONE TOPOLOGICAL PROPERTIES	7
1 General Topology	9
1.1 Topological Spaces	9
1.2 Compactness	15
2 Metric Spaces	18
2.1 Metric Spaces	18
2.2 The Topology of Metric Spaces	21
2.3 Completeness: Tietze's Extension Theorem	24
2.4 More on Completeness	27
2.5 The Completion of a Metric Space	29
2.6 Topologically Complete Spaces	31
2.7 Baire's Category Theorem	33
2.8 Lipschitz Functions	35
3 Polish Spaces and Compactness	38
3.1 Polish Spaces	38
3.2 Totally Bounded Metric Spaces	39
3.3 Compact Metrizable Spaces	41
3.4 Locally Compact Polish Spaces	47
4 Semi-continuous Functions	50
4.1 The Effective Domain and Proper Functions	50
4.2 Semi-continuity	50
4.3 The Brézis–Browder Lemma	53
4.4 Ekeland's Variational Principle	54

5	Uniform Spaces and Topological Groups	56
5.1	Uniform Spaces	56
5.2	The Uniformity of a Compact Hausdorff Space	59
5.3	Topological Groups	61
5.4	The Uniformities of a Topological Group	64
5.5	Group Actions	66
5.6	Metrizable Topological Groups	67
6	Càdlàg Functions	71
6.1	Càdlàg Functions	71
6.2	The Space $(D[0, 1], d_\infty)$	72
6.3	The Skorohod Topology	73
6.4	The Metric d_B	75
7	Banach Spaces	79
7.1	Normed Spaces and Banach Spaces	79
7.2	The Space $BL(X)$ of Bounded Lipschitz Functions	82
7.3	Introduction to Convexity	83
7.4	Convex Sets in a Normed Space	86
7.5	Linear Operators	88
7.6	Five Fundamental Theorems	91
7.7	The Petal Theorem and Daneš's Drop Theorem	95
8	Hilbert Spaces	97
8.1	Inner-product Spaces	97
8.2	Hilbert Space; Nearest Points	101
8.3	Orthonormal Sequences; Gram–Schmidt Orthonormalization	104
8.4	Orthonormal Bases	107
8.5	The Fréchet–Riesz Representation Theorem; Adjoints	108
9	The Hahn–Banach Theorem	112
9.1	The Hahn–Banach Extension Theorem	112
9.2	The Separation Theorem	116
9.3	Weak Topologies	118
9.4	Polarity	119
9.5	Weak and Weak* Topologies for Normed Spaces	120
9.6	Banach's Theorem and the Banach–Alaoglu Theorem	124
9.7	The Complex Hahn–Banach Theorem	125

10	Convex Functions	128
10.1	Convex Envelopes	128
10.2	Continuous Convex Functions	130
11	Subdifferentials and the Legendre Transform	133
11.1	Differentials and Subdifferentials	133
11.2	The Legendre Transform	134
11.3	Some Examples of Legendre Transforms	137
11.4	The Episum	139
11.5	The Subdifferential of a Very Regular Convex Function	140
11.6	Smoothness	143
11.7	The Fenchel–Rockafeller Duality Theorem	148
11.8	The Bishop–Phelps Theorem	149
11.9	Monotone and Cyclically Monotone Sets	151
12	Compact Convex Polish Spaces	155
12.1	Compact Polish Subsets of a Dual Pair	155
12.2	Extreme Points	157
12.3	Dentability	160
13	Some Fixed Point Theorems	162
13.1	The Contraction Mapping Theorem	162
13.2	Fixed Point Theorems of Caristi and Clarke	165
13.3	Simplices	167
13.4	Sperner’s Lemma	168
13.5	Brouwer’s Fixed Point Theorem	170
13.6	Schauder’s Fixed Point Theorem	171
13.7	Fixed Point Theorems of Markov and Kakutani	173
13.8	The Ryll–Nardzewski Fixed Point Theorem	175
	PART TWO MEASURES ON POLISH SPACES	177
14	Abstract Measure Theory	179
14.1	Measurable Sets and Functions	179
14.2	Measure Spaces	182
14.3	Convergence of Measurable Functions	184
14.4	Integration	187
14.5	Integrable Functions	188

15 Further Measure Theory	191
15.1 Riesz Spaces	191
15.2 Signed Measures	194
15.3 $M(X)$, L^1 and L^∞	196
15.4 The Radon–Nikodym Theorem	199
15.5 Orlicz Spaces and L^p Spaces	203
16 Borel Measures	210
16.1 Borel Measures, Regularity and Tightness	210
16.2 Radon Measures	214
16.3 Borel Measures on Polish Spaces	215
16.4 Lusin’s Theorem	216
16.5 Measures on the Bernoulli Sequence Space $\Omega(\mathbf{N})$	218
16.6 The Riesz Representation Theorem	222
16.7 The Locally Compact Riesz Representation Theorem	225
16.8 The Stone–Weierstrass Theorem	226
16.9 Product Measures	228
16.10 Disintegration of Measures	231
16.11 The Gluing Lemma	234
16.12 Haar Measure on Compact Metrizable Groups	236
16.13 Haar Measure on Locally Compact Polish Topological Groups	238
17 Measures on Euclidean Space	243
17.1 Borel Measures on \mathbf{R} and \mathbf{R}^d	243
17.2 Functions of Bounded Variation	245
17.3 Spherical Derivatives	247
17.4 The Lebesgue Differentiation Theorem	249
17.5 Differentiating Singular Measures	250
17.6 Differentiating Functions in bv_0	251
17.7 Rademacher’s Theorem	254
18 Convergence of Measures	257
18.1 The Norm $\ \cdot\ _{TV}$	257
18.2 The Weak Topology w	258
18.3 The Portmanteau Theorem	260
18.4 Uniform Tightness	264
18.5 The β Metric	266
18.6 The Prokhorov Metric	269
18.7 The Fourier Transform and the Central Limit Theorem	271
18.8 Uniform Integrability	276
18.9 Uniform Integrability in Orlicz Spaces	278

19	Introduction to Choquet Theory	280
19.1	Barycentres	280
19.2	The Lower Convex Envelope Revisited	282
19.3	Choquet's Theorem	284
19.4	Boundaries	285
19.5	Peak Points	289
19.6	The Choquet Ordering	291
19.7	Dilations	293
	 PART THREE INTRODUCTION TO OPTIMAL TRANSPORTATION	 297
20	Optimal Transportation	299
20.1	The Monge Problem	299
20.2	The Kantorovich Problem	300
20.3	The Kantorovich–Rubinstein Theorem	303
20.4	c -concavity	305
20.5	c -cyclical Monotonicity	308
20.6	Optimal Transport Plans Revisited	310
20.7	Approximation	313
21	Wasserstein Metrics	315
21.1	The Wasserstein Metrics W_p	315
21.2	The Wasserstein Metric W_1	317
21.3	W_1 Compactness	318
21.4	W_p Compactness	320
21.5	W_p -Completeness	322
21.6	The Mallows Distances	323
22	Some Examples	325
22.1	Strictly Subadditive Metric Cost Functions	325
22.2	The Real Line	326
22.3	The Quadratic Cost Function	327
22.4	The Monge Problem on \mathbf{R}^d	329
22.5	Strictly Convex Translation Invariant Costs on \mathbf{R}^d	331
22.6	Some Strictly Concave Translation–Invariant Costs on \mathbf{R}^d	336
	 <i>Further Reading</i>	 339
	<i>Index</i>	342

Introduction

Analysis is concerned with continuity and convergence. Investigation of these ideas led to the notions of topology and topological spaces. Once these had been introduced, they became subjects in their own right, which were investigated in fine detail to see how far the theory might lead (an excellent illustration of this is given by the fascinating book by Steen and Seebach [SS]).

In practice, however, a great deal of analysis is concerned with what happens on a very restricted class of topological spaces, namely, the Polish spaces. A Polish space is a separable topological space whose topology is defined by a complete metric. Important examples include Euclidean space, pathwise-connected Riemannian manifolds, compact metric spaces and separable Banach spaces.

The purpose of this book is to develop the study of analysis on Polish spaces. It consists of three parts. The first considers topological properties of Polish spaces, and the second deals with the theory of measures on Polish spaces. In the third part, we give an introduction to the theory of optimal transportation. This makes essential use of the results of the first two parts, or modifications of them. It was, in fact, study of optimal transportation that led to the realization of how much its study required properties of Polish spaces, and measures on them.

There are three important advantages of restricting attention to Polish spaces. First, many of the curious complications of the general topological theory disappear. For example, a subspace of a separable topological space need not be separable, whereas a subspace of a separable metric space is always separable. Secondly, the proofs of standard results are frequently much easier in this restricted setting. For example, Urysohn's lemma for normal topological spaces is quite delicate, whereas it is very easy for metric spaces. Thirdly, Polish spaces enjoy some very important properties. Thus it follows from Alexandroff's theorem that a topological space is a Polish space if and only

if it is homeomorphic to a G_δ subset of the Hilbert cube $\mathbf{H} = [0, 1]^{\mathbf{N}}$, which is a compact metrizable space. From this, or directly, it follows that a Borel measure on a Polish space is tight (Ulam's theorem: the measure of a Borel set can be approximated from below by the measures of compact sets contained in it). It also means that we can push forward a Borel measure on a Polish space X to a Borel measure on a compact metric space containing X . This greatly simplifies both the measure theory and also the construction of measures. In fact, I believe that almost all the probability measures that arise in practice are Borel measures on Polish spaces; one important exception, which we do not consider or need, is the theory of uniform central limit theorems.

One major advantage of restricting attention to Polish spaces is that it is not necessary to appeal to the axiom of choice. Instead, we proceed by induction, using the axiom of dependent choice; we make an infinite sequence of decisions, each possibly dependent on what has gone before.

In analysis, there are a few fundamental results which require the axiom of choice. The first is Tychonoff's theorem, which states that an arbitrary product of compact topological spaces, with the product topology, is compact. We do not prove this, or use it. On the other hand, we do prove, and use, the fact that a countable product of compact metrizable spaces is compact and metrizable.

Secondly, there are two fundamental results of linear analysis which need the axiom of choice, using Zorn's lemma. The first of these is the Hahn–Banach theorem (together with the separation theorem). Using induction, we prove weak forms of these, for separable normed spaces; this is sufficient for our purposes.

But for completeness' sake we also give the classical results, using Zorn's lemma; Here we first prove the separation theorem, showing that it essentially depends upon the connectedness of the unit circle \mathbf{T} , and then derive the Hahn–Banach theorem from it.

The other fundamental result which requires the axiom of choice is the Krein–Mil'man theorem, which states that every weakly compact convex subset K has an extreme point. Again, we only need, and use, the result in the case where K is metrizable, and we prove this without the axiom of choice.

The fact that we avoid using the axiom of choice suggests that the proofs should, in some sense, be less abstract and more constructive. Unfortunately, this is not the case; the arguments that are used are frequently indirect (consider the collection of all sets with a particular property), so that for example a typical Borel subset of a Polish space does not have a simple description.

Let us now describe the contents of the three parts of this book in more detail.

Part I: Topological Properties

Although it is assumed that the reader has some knowledge of general topology and metric spaces, the first two chapters give an account of these topics, including Tietze's extension theorem, Baire's category theorem and Lipschitz functions.

This leads to the notion of a Polish space, a separable topological space whose topology is given by a complete metric. A fundamental example is given by a compact metrizable space, and Alexandroff's theorem is used to show that a topological space is a Polish space if and only if it is homeomorphic to a G_δ subspace of a compact metric space, and in particular homeomorphic to a G_δ subspace of the Hilbert cube.

We shall need to consider suprema of sets of real-valued continuous functions. Such functions are lower semi-continuous, and we consider such functions in Chapter 4. A lower semi-continuous function on a compact space attains its infimum, but this is not necessarily true for lower semi-continuous functions on a complete metric space. We establish its replacement, Ekeland's variational principle, together with two of its corollaries, the petal theorem and Daneš's drop theorem, and various other applications.

Metric spaces have more structure than a topological one, and Chapter 5 contains an account of uniform spaces; uniformity is particularly important when we consider locally compact topological groups, in Part II.

Chapter 6 is devoted to showing that the space of càdlàg functions is a Polish space under the Skorohod topology; many stochastic processes, and their underlying measures, lie on such spaces, and this helps justify the claim that almost all probability measures of interest lie on Polish spaces. Further examples are given by separable Banach spaces and Hilbert spaces; these are principally used to introduce the notion of convexity.

The rest of Part I is concerned with convexity. The Hahn–Banach theorem is one of the key results here, and we give proofs of appropriate results, both without and with the axiom of choice. For us, the Hahn–Banach theorem is essentially a geometric theorem showing that two suitable convex sets can be separated by a hyperplane. It also leads onto the notion of weak topology.

The Legendre transform provides an important duality theory for convex functions, and this leads naturally to the concept of subdifferentials and subdifferentiability. We prove the Bishop–Phelps theorem, and also introduce the notion of cyclic monotonicity.

The rest of Part I is concerned with convex sets which are compact and metrizable in some suitable topology. We prove versions of the Krein–Mil'man theorem, Krein's theorem and a swathe of fixed point theorems, many of which are used later.

Part II: Measures on Polish Spaces

We expect that the reader has some knowledge of abstract measure theory, but Chapter 14 contains a survey of the basic results. Chapter 15 contains some further results: we introduce the Banach space $M(X)$ of finite measures on a Polish space X , its subspaces $L^1(\mu)$ and Orlicz spaces (with the use of Legendre duality). We give von Neumann's Hilbert space proof of the Radon–Nikodym property and a proof of the strong law of large numbers (to be used later).

In Chapter 16, we investigate Borel measures on Polish spaces. We prove regularity and tightness properties; we may not know what a typical Borel set looks like, but we can approximate the Borel measure of a Borel set from the outside by open sets, and on the inside by compact sets. This leads to Lusin's theorem, which says that if μ is a Borel measure on a Polish space X then a Borel measurable function on X is continuous on a large compact subset.

So far, all is theory, and no measures, other than trivial ones, have been shown to exist. We remedy this by showing how to construct Borel measures on the Bernoulli space $\Omega(\mathbb{N})$, and then, pushing forward, constructing measures on compact metric spaces and Polish spaces. We prove the Riesz representation theorem, and use this to give a measure-theoretic proof of the Stone–Weierstrass theorem.

We then show how Borel measures can be disintegrated, and establish the existence of Haar measure on compact and locally compact Polish spaces; we follow an account by Pedersen to show that this last result is relatively straightforward.

In Chapter 17, we come down to earth and consider Borel measures on Euclidean space, where the point at issue is the differentiation of measures and of Borel measurable functions. We establish Lebesgue's differentiation theorem and Rademacher's theorem on the differentiability almost everywhere of Lipschitz functions.

We now proceed to study one of the key points of this chapter, namely, the weak convergence of measures. We show that there are various metrics which define the weak topology w , and show that although the unit ball $M_1(X)$ is generally not metrizable, the space of probability measures $P(X)$ is a Polish space. Examples of weak convergence include the central limit theorem and the empirical law of large numbers. Finally, uniform integrability is investigated.

Part II ends with an introduction to Choquet theory on a metrizable compact convex set. The theory is notoriously difficult for general weakly compact convex sets, but the difficulties disappear in the metrizable case.

Parts I and II contain more than two hundred exercises. These are usually very straightforward, but most are an essential part of the text; do them.