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Noether's Theorems

Applications in Mechanics and Field Theory

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To my wife
Aida Karamysheva
Professor, molecule biologist

Preface

Noether's first and second theorems are formulated in a very general setting of reducible degenerate Grassmann-graded Lagrangian theory of even and odd variables on graded bundles.

Lagrangian theory generally is characterized by a hierarchy of nontrivial Noether and higher-stage Noether identities and the corresponding gauge and higher-stage gauge symmetries which characterize the degeneracy of a Lagrangian system. By analogy with Noether identities of differential operators, they are described in the homology terms. In these terms, Noether's inverse and direct second theorems associate to the Koszul–Tate graded chain complex of Noether and higher-stage Noether identities the gauge cochain sequence whose ascent gauge operator provides gauge and higher-stage gauge symmetries of Grassmann-graded Lagrangian theory. If these symmetries are algebraically closed, an ascent gauge operator is generalized to a nilpotent BRST operator which brings a gauge cochain sequence into a BRST complex and provides the BRST extension of original Lagrangian theory.

In the present book, the calculus of variations and Lagrangian formalism are phrased in algebraic terms of a variational bicomplex on an infinite order jet manifold that enables one to extend this formalism to Grassmann-graded Lagrangian systems of even and odd variables on graded bundles. Cohomology of a graded variational bicomplex provides the global solutions of the direct and inverse problems of the calculus of variations.

In this framework, Noether's direct first theorem is formulated as a straightforward corollary of the global variational formula. It associates to any Lagrangian symmetry the conserved symmetry current whose total differential vanishes on-shell. Proved in a very general setting, so-called Noether's third theorem states that a conserved symmetry current along any gauge symmetry is reduced to a superpotential, i.e., it is a total differential on-shell. This also is the case of covariant Hamiltonian formalism on smooth fibre bundles seen as the particular Lagrangian one on phase Legendre bundles.

Lagrangian formalism on smooth fibre bundles and graded bundles provides the comprehensive formulations both of classical field theory and nonrelativistic mechanics.

Non-autonomous nonrelativistic mechanics is adequately formulated as particular Lagrangian and Hamiltonian theory on a configuration bundle over the time axis. Conserved symmetry currents of Noether's first theorem in mechanics are integrals of motion, but the converse need not be true. In Hamiltonian mechanics, Noether's inverse first theorem states that all integrals of motion come from symmetries. In particular, this is the case of energy functions with respect to different reference frames.

The book presents a number of physically relevant models: superintegrable Hamiltonian systems, the global Kepler problem, Yang–Mills gauge theory on principal bundles, SUSY gauge theory, gauge gravitation theory on natural bundles, topological Chern–Simons field theory and topological BF theory, exemplifying a reducible degenerate Lagrangian system.

Our book addresses to a wide audience of theoreticians, mathematical physicists and mathematicians. With respect to mathematical prerequisites, the reader is expected to be familiar with the basics of differential geometry of fibre bundles. We have tried to give the necessary mathematical background, thus making our exposition self-contained. For the sake of convenience of the reader, a number of relevant mathematical topics are compiled in appendixes.

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Introduction

Noether's theorems are well known to treat symmetries of Lagrangian systems. Noether's first theorem associates to a Lagrangian symmetry the conserved symmetry current whose total differential vanishes on-shell. The second ones provide the correspondence between Noether identities and gauge symmetries of a Lagrangian system. We refer the reader to the brilliant book of Yvette Kosmann-Schwarzbach [84] for the history and references on this subject.

Our book aims to present Noether's theorems in a very general setting of reducible degenerate Grassmann-graded Lagrangian theory of even and odd variables on graded bundles.

We however start with even Lagrangian formalism on smooth fibre bundles (Chap. 2), and focus ourselves especially on first-order Lagrangian theory (Chap. 3) because the most physically relevant models (nonrelativistic mechanics, gauge field theory, gravitation theory, etc.) are of this type.

Lagrangian theory of even (commutative) variables on an n-dimensional smooth manifold X conventionally is formulated in terms of fibre bundles over X and jet manifolds of their sections [15, 53, 108, 133, 143], and it lies in the framework of general technique of nonlinear differential equations and operators [25, 53, 85]. This formulation is based on the categorial equivalence of projective $C^\infty(X)$-modules of finite rank and vector bundles over X in accordance with the classical Serre–Swan theorem, generalized to noncompact manifolds (Theorem A.10).

The calculus of variations and Lagrangian formalism on a fibre bundle $Y \to X$ can be adequately formulated in algebraic terms of the variational bicomplex (1.21) of differential forms on an infinite order jet manifold $J^\infty Y$ of sections of $Y \to X$ [3, 15, 56, 61, 108, 133, 143]. In this framework, finite order Lagrangians and Euler–Lagrange operator are defined as elements (1.32)–(1.33) of this bicomplex (Sect. 1.3). The cohomology of the variational bicomplex (Sect. 1.2) provides a solution of the global inverse problem of the calculus of variations (Theorems 1.15–1.16), and states the global variational formula (1.36) for Lagrangians and Euler–Lagrange operators (Theorem 1.17).

In these terms, Noether's first theorem (Theorem 2.7) and Noether's direct second theorem (Theorem 2.6) are straightforward corollaries of the global decomposition (1.36).

Noether's first theorem associates to any symmetry of a Lagrangian L (Definition 2.7) the conserved symmetry current (2.21) whose total differential vanishes on-shell (Sect. 2.3). One can show that a conserved symmetry current itself is a total differential on-shell if it is associated to a gauge symmetry (Theorem 2.8).

Treating gauge symmetries of Lagrangian theory, one is traditionally based on an example of Yang–Mills gauge theory of principal connections on the principal bundle $P \to X$ (8.3) where gauge symmetries are vertical principal automorphisms of this bundle (Sect. 8.2). They are represented by global sections of the associated group bundle (8.43) and, thus, look like symmetries depending on parameter functions. This notion of gauge symmetries is generalized to Lagrangian theory on an arbitrary fibre bundle $Y \to X$ (Definition 2.3) and on graded bundles (Definition 7.3).

Given a gauge symmetry of a Lagrangian system on fibre bundles, Noether's direct second theorem (Theorem 2.6) states that its Euler–Lagrange operator obeys the corresponding Noether identities (2.17) (Sect. 2.2). A problem is that any Euler–Lagrange operator satisfies Noether identities, which therefore must be separated into the trivial and nontrivial ones. These Noether identities can obey first-stage Noether identities, which in turn are subject to the second-stage ones and so on. Thus, there is a hierarchy of nontrivial Noether and higher-stage Noether identities which characterizes the degeneracy of a Lagrangian theory (Sect. 7.1). A Lagrangian system is called degenerate if it admits nontrivial Noether identities and reducible if there exist nontrivial higher-stage Noether identities. We follow the general analysis of Noether and higher-stage Noether identities of differential operators on fibre bundles when trivial and nontrivial Noether identities are described by boundaries and cycles of a certain chain complex [61, 123]. This description involves Grassmann-graded objects.

In a general setting, we therefore consider Grassmann-graded Lagrangian systems of even and odd variables (Chap. 6).

Different geometric models of odd variables either on graded manifolds or supermanifolds are discussed [28, 30, 46, 107, 134]. Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras [7, 61, 131]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. We follow the Serre–Swan theorem for graded manifolds (Theorem 6.2) [11, 61]. It states that, if a graded commutative $C^\infty(X)$-ring is generated by a projective $C^\infty(X)$-module of finite rank, it is isomorphic to a ring of graded functions on a graded manifold whose body is a smooth manifold X. Accordingly, we describe odd variables on a smooth manifold X in terms of graded bundles over X [61, 134].

Let us recall that a graded manifold is a local-ringed space (Definition C.1), characterized by a smooth body manifold Z and some structure sheaf $\mathfrak{A}$ of Grassmann algebras on Z [7, 61, 134]. Its sections form a graded commutative

$C^\infty(Z)$-ring $\mathscr{A}$ of graded functions on a graded manifold $(Z, \mathfrak{A})$. It is called the structure ring of $(Z, \mathfrak{A})$. The differential calculus on a graded manifold is defined as the Chevalley–Eilenberg differential calculus over its structure ring (Sect. 6.2). By virtue of the well-known Batchelor theorem (Theorem 6.1), there exists a vector bundle $E \to Z$ with a typical fibre V such that the structure sheaf $\mathfrak{A}$ of $(Z, \mathfrak{A})$ is isomorphic to a sheaf $\mathfrak{A}_E$ of germs of sections of the exterior bundle $\wedge E^*$ of the dual E^* of E whose typical fibre is the Grassmann algebra $\wedge V^*$ [7, 14]. This Batchelor's isomorphism is not canonical. In applications, it however is fixed from the beginning. Therefore, we restrict our consideration to graded manifolds $(Z, \mathfrak{A}_E)$, termed the simple graded manifolds, modelled over vector bundles $E \to Z$ (Definition 6.4).

Let us note that a manifold Z itself can be treated as a trivial simple graded manifold (Z, C_Z^∞) modelled over a trivial fibre bundle $Z \times \mathbb{R} \to Z$ whose structure ring of graded functions is reduced to a commutative ring $C^\infty(Z)$ of smooth real functions on Z (Remark 6.1). Accordingly, a configuration fibre bundle $Y \to X$ in Lagrangian theory of even variables can be regarded as the graded bundle (6.29) of trivial graded manifolds (Remark 6.4). It follows that, in a general setting, one can define a configuration space of Grassmann-graded Lagrangian theory of even and odd variables as being the graded bundle $(X, Y, \mathfrak{A}_F)$ (6.32) over a trivial graded manifold (X, C_X^∞) modelled over the smooth composite bundle $F \to Y \to X$ (6.33) (Sect. 6.5). If $Y \to X$ is a vector bundle, this is the particular case of graded vector bundles in [73, 107] whose base is a trivial graded manifold.

By analogy with the calculus of variations and Lagrangian theory on smooth fibre bundles, Grassmann-graded Lagrangian theory on a graded bundle $(X, Y, \mathfrak{A}_F)$ (Sect. 6.5) comprehensively is phrased in terms of the Grassmann-graded variational bicomplex (6.54) of graded exterior forms on a graded infinite order jet manifold $(J^\infty Y, \mathfrak{A}_{J^\infty F})$ [6, 11, 12, 61, 134, 137]. Graded Lagrangians and Euler–Lagrange operator are defined as elements (6.69) and (6.70) of this graded bicomplex. The cohomology of a Grassmann-graded variational bicomplex (Theorems 6.9–6.10) provides a solution of the global inverse problem of the calculus of variations (Theorem 6.11), and states the global variational formula (6.73) for graded Lagrangians and Euler–Lagrange operators (Theorem 6.13).

In these terms, Noether's first theorem is formulated in a very general setting as a straightforward corollary of the global variational formula (6.73) (Sect. 6.6). It associates to any supersymmetry of a graded Lagrangian L (Definition 6.9) the conserved supersymmetry current (6.84) whose graded total differential vanishes on-shell (Theorem 6.20). One can show that a conserved supersymmetry current along a gauge supersymmetry is a graded total differential on-shell (Theorem 7.11).

Given a gauge supersymmetry of a graded Lagrangian, Noether's direct second theorem (Theorem 7.10) states that an Euler–Lagrange operator obeys the corresponding Noether identities. As was mentioned above, a problem is that any Euler–Lagrange operator satisfies Noether identities, which therefore must be separated into the trivial and nontrivial ones, and that there is a hierarchy of Noether and higher-stage Noether identities.

We follow the general analysis of Noether identities and higher-stage Noether identities of differential operators on fibre bundles (Appendix D). If a certain homology regularity condition (Definition 7.2) holds, one can associate to a Grassmann-graded Lagrangian system the exact Koszul–Tate chain complex (7.28) possessing the boundary Koszul–Tate operator (7.26) whose nilpotentness is equivalent to all complete nontrivial Noether identities (7.12) and higher-stage Noether identities (7.29) (Theorem 7.5) [11, 12, 61, 134, 137].

It should be noted that the notion of higher-stage Noether identities has come from that of reducible constraints. The Koszul–Tate complex of Noether identities has been invented similarly to that of constraints under the regularity condition that Noether identities are locally separated into the independent and dependent ones [6, 44]. This condition is relevant for constraints, defined by a finite set of functions which the inverse mapping theorem is applied to. However, Noether identities of differential operators, unlike constraints, are differential equations (Appendix D). They are given by an infinite set of functions on a Fréchet manifold of infinite order jets where the inverse mapping theorem fails to be valid. Therefore, the regularity condition for the Koszul–Tate complex of constraints is replaced with the above mentioned homology regularity condition.

Noether's inverse second theorem formulated in homology terms (Theorem 7.9) associates to the Koszul–Tate chain complex (7.28) the cochain sequence (7.38) with the ascent operator (7.39), called the gauge operator, whose components are complete nontrivial gauge and higher-stage gauge supersymmetries of Grassmann-graded Lagrangian theory [12, 61, 134, 137].

The gauge operator unlike the Koszul–Tate one is not nilpotent, unless gauge symmetries are Abelian (Remark 7.12). Therefore, an intrinsic definition of nontrivial gauge and higher-stage gauge symmetries meets difficulties. Another problem is that gauge symmetries need not form an algebra [48, 60, 63]. However, we can say that gauge symmetries are algebraically closed in a sense if the gauge operator admits a nilpotent extension, termed the BRST (Becchi–Rouet–Stora–Tyitin) operator (Sect. 7.5). If the BRST operator exists, the above mentioned cochain sequence is brought into the BRST complex. The Koszul–Tate and BRST complexes provide a BRST extension of original Lagrangian theory by Grassmann-graded ghosts and Noether antifields.

This extension is a preliminary step towards the BV (Batalin–Vilkovisky) quantization of reducible degenerate Lagrangian theories [6, 13, 44, 50, 63].

For the purpose of applications in field theory and mechanics, Chap. 3 of the book addresses first-order Lagrangian and covariant Hamiltonian theories on fibre bundles since the most of relevant field models is of this type. We restrict our consideration to classical symmetries represented by the projectable vector fields u (3.24) on a configuration bundle $Y \to X$ (Sect. 3.3). Since the corresponding symmetry current (3.29) is linear in a vector field u, one usually deals with the following types of symmetries.

(i) If u is a vertical vector field, the corresponding symmetry current is the Noether one (3.32). This is the case of so-called internal symmetries

(ii) Let τ be a vector field on X. Then its lift $\gamma\tau$ (B.27) onto Y (Definition B.2) provides the corresponding energy-momentum current (3.33) (Definition 3.3). We usually have either the horizontal lift $\Gamma\tau$ (B.56) by means of a connection Γ on $Y \to X$, e.g., in Yang–Mills gauge theory (Sect. 8.4), or the functorial lift $\widetilde{\tau}$ (Definition B.28) which is an infinitesimal general covariant transformation. This is the case of gauge gravitation theory (Sect. 10.4).

Applied to field theory, the familiar symplectic Hamiltonian technique takes a form of instantaneous Hamiltonian formalism on an infinite-dimensional phase space, where canonical variables are field functions at each instant of time [65]. The true Hamiltonian counterpart of first-order Lagrangian theory on a fibre bundle $Y \to X$ is covariant Hamiltonian formalism, where canonical momenta p_i^μ correspond to jets y_μ^i of field variables y^i with respect to all base coordinates x^μ. This formalism has been vigorously developed since 1970s in the Hamilton–De Donder, polysymplectic, multisymplectic, k-symplectic, k-cosymplectic and other variants [26, 27, 36, 45, 64, 69, 72, 81, 92, 95, 96, 103, 112, 114, 117]. We follow polysymplectic Hamiltonian formalism on a fibre bundle $Y \to X$ where the Legendre bundle Π (3.6) plays the role of a phase space (Sect. 3.5). A key point is that polysymplectic Hamiltonian formalism on a phase space Π is equivalent to particular first-order Lagrangian theory on a configuration space $\Pi \to X$. This fact enables us to describe symmetries of Hamiltonian field theory similarly to those in Lagrangian formalism (Sect. 3.7).

One can formulate non-autonomous nonrelativistic mechanics as particular field theory on fibre bundles $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ [62, 98, 121, 135]. Its velocity space is the first-order jet manifold J^1Q of sections of a configuration bundle $Q \to \mathbb{R}$, and its phase space Π (3.8) is the vertical cotangent bundle V^*Q of $Q \to \mathbb{R}$ (Chap. 4). A difference between mechanics and field theory however lies in the fact that fibre bundles over $\mathbb{R}$ always are trivial, and that all connections on these fibre bundles are flat. Consequently, they are not dynamic variables, but characterize nonrelativistic reference frames (Definition 4.1). By virtue of Noether's first theorem, any symmetry defines a symmetry current which also is an integral of motion in Lagrangian and Hamiltonian mechanics (Theorem 4.4). The converse is not true in Lagrangian mechanics where integrals of motion need not come from symmetries. We show that, in Hamiltonian mechanics, any integral of motion is a symmetry current (Theorem 4.15). One can think of this fact as being Noether's inverse first theorem.

The book presents a number of physically relevant models: commutative and noncommutative integrable Hamiltonian systems (Sect. 4.7), the global Kepler problem (Chap. 5), Yang–Mills gauge theory on principal bundles (Chap. 8), SUSY gauge theory (Chap. 9) on principal graded bundles, gauge gravitation theory on natural bundles (Chap. 10), topological Chern–Simons field theory (Chap. 11), and topological BF theory, exemplifying a reducible degenerate Lagrangian system (Chap. 12).

For the sake of convenience of the reader, a number of relevant mathematical topics are compiled in appendixes, thus making the exposition self-contained.

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