Functional Analysis

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A Wiley-Interscience Publication

175 YEARS OF PUBLISHING
1807 1982

JOHN WILEY & SONS
Chichester · New York · Brisbane · Toronto · Singapore

This edition is published by permission of Verlag B G Teubner, Stuttgart, and is the sole authorized English translation of the original German edition.

British Library Cataloguing in Publication Data:

Heuser, Harro
Functional analysis.
1. Functional analysis
I. Title II. Funktionanalysis. *English*
515.7 QA320

ISBN 0 471 28052 6 (cloth)
0 471 10069 2 (paper)

Filmset by Composition House Limited,
Salisbury, Wiltshire, England.

Printed in the United States of America.

Preface

A science as vitally important as functional analysis can fortunately not be defined. But its great 'Leitmotiv' can nevertheless be indicated: It is the fusion of algebraic and topological structures. This seemingly abstract and anaemic subject has developed in such a rich and lively manner that nowadays functional analysis has a strong influence on a great number of completely different fields inside and outside of mathematics: systems engineers and atomic physicists cannot do without it, just like mathematicians working in numerical analysis, differential or integral equations, the theory of approximation or representation theory—just to name a few.

The purpose of this book is to give a lively and thorough exposition of the basic concepts, the essential statements, the main methods and finally also of the way of thinking of functional analysis and to satisfy the needs of a large circle of users, in content as well as didactically.

To achieve this purpose I chose an orientation toward problems. The present book starts out, whenever possible, from questions and facts of classical analysis and algebra, and tries to get to their core by leaving aside what is accidental in order to obtain general concepts and assertions. Conversely, it aims at making the newly acquired tools fruitful for the classical fields. Naturally in the course of this process so many questions of a 'purely functional analytic nature' accumulate that finally functional analysis is propelled by its own problems. But the main text and the exercises return constantly to the familiar world of analysis and algebra. I hope to enhance in this way the motivation and the intuitive understanding of the reader and to save him from that particular feeling of being lost, which occurs so easily and annoyingly when studying an abstract theory. I think that in this way I also acquit myself best of the duty imposed on an author by the word 'Leitfaden'[1], i.e., leading thread. The first leading thread was that of Ariadne; Plutarch reports about it in his biography of Theseus the following[2]: '. . . after he (Theseus) arrived in Creta, he slew the

[1] The original German edition was published in the 'Mathematische Leitfaden' series of B. G. Teubner.

[2] Plutarch: The Lives of the Noble Grecians and Romans, translated by Thomas North, the Nonesuch Press, London, 1929.

Minotaur . . . by the means and help of Ariadne, who being fallen in fancy with him, did give him a clue of thread, by the help whereof she taught him how he might easily wind out of the turnings and cranks of the Labyrinth'.

An organization directed towards problems does not try to represent a science as it evolved—this is the purpose of the genetic method—but how it could also have evolved. It is the coming together of fortunate circumstances—maybe even more—that one of the foundations of this book, the concept of a bilinear system, already is tacitly the basis of the pioneering works of Fredholm concerning integral equations, which started functional analysis. The problem of the solvability of Fredholm's integral equations will therefore play a central role in this book. It leads us through the Neumann series to the theory of Banach algebras and through the concept of a bilinear system and of normal solvability to the extension principle of Hahn–Banach and to the duality theory which follows from it, a crown jewel of functional analysis. It should be observed emphatically that the investigation of Fredholm's integral equation originates from a very concrete situation: many boundary value problems of physics and of technology can be transformed into an equation of this kind.

The table of contents gives detailed information about the subjects treated and their interdependence. I want to indicate here only a few major blocks. The first seven chapters are dominated by the problem of equations: under what conditions are equations in general spaces solvable, how can they be effectively solved, and how do the solutions depend on certain initial data? Mainly linear problems will be considered; the non-linear domain will be represented by the fixed-point theorems of Banach and Schauder; the latter occur, however, first only in §106. In Chapter VIII approximation problems will arise. The great subject of orthogonality will emerge here, and will be developed in the following two chapters (orthogonal decomposition and spectral theory in Hilbert spaces). Chapter XI to XIII reach the summit of the progress towards abstraction: they show how linear and topological structures are fusioned in the concept of a topological vector space. This fusion leads, if the structures are rich enough, back to a quite concrete situation: commutative, complex B^*-algebras are nothing else but algebras of continuous functions. With this theorem of Gelfand and Neumark the book concludes.

The monumental work of Dunford and Schwartz 'Linear Operators' has 2592 pages; the theory of topological vector spaces is only touched in it. It is needless to say that my book had to sacrifice many important or only attractive subjects. I did not want to sacrifice, however, a copious motivation, an illustration from several angles of the central facts, and the details of the proofs.

For long stretches only the elements of analysis and linear algebra are required as prerequisites. The concept of a metric space and its continuous maps will be developed in the book. I listed in §81 and §101 without proofs the few topological facts which will be needed from §82 on. The spectral theory in Chapter VII cannot be studied profitably without some knowledge of the theory of analytic functions of a complex variable: one needs Cauchy's integral theorem, Liouville's theorem, the Taylor and the Laurent expansions. At a few places theorems from real and complex analysis will be used which might not be familiar to every reader (e.g., the Stone–Weierstrass theorem); in these cases I have indicated references to the literature, where the proofs can easily be read. The Lebesgue integral and the theory of partial differential equations will not be used.

I have to thank cordially the Universidad de los Andes in Bogotá (Colombia) and the University of Toronto in Toronto (Canada): they gave me through visiting positions the possibility to work intensively on this book. Mrs. Mia Münzel put her home, which lies quietly above Lake Garda, at my disposition for the last chapters; I am deeply indebted to her. My special thanks go to Dr. H. Kroh, Dr. U. Mertins and Mr. G. Schneider, further to Mrs. Y. Paasche and Mrs. K. Zeder. The three gentlemen were at my side during the preparation of this book with advice and help, they improved it and cleaned it up through valuable indications, and read all the proofs; the two ladies transformed with unbelievable patience a miserable manuscript into a clean typescript. I thank the Teubner–Verlag cordially for the pleasant collaboration and for its watchful help and assistance.

Nastätten in Taunus, August 1975 HARRO HEUSER

List of symbols

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Introduction

In this section some notation and facts will be listed, which are of fundamental importance for everything that follows.

General notation

N, **R**, **C** denote the set of natural, real and complex numbers, respectively. **K** stands for either the field **R** or the field **C**; elements of **K** are also called *scalars* and they will usually be denoted by lower case Greek characters. Re α and Im α denote the real and the imaginary part of α, respectively. $\bar{\alpha}$ is the conjugate of the complex number α. We denote by $|\alpha_{\nu\mu}|$ the determinant of the n by n matrix $(\alpha_{\nu\mu})$; there is no risk of confusion with the absolute value of the number $\alpha_{\nu\mu}$. In equations which are definitions we use the symbols ':=' or '=:', where the colon stands on the side of the symbol which is to be defined.

Examples: 1. $f(x) := x^2$; 2. $\{1, 2, 3\} =: M$; 3. *Kronecker's symbol*

$$\delta_{ik} := \begin{cases} 1 & \text{for} \quad i = k \\ 0 & \text{for} \quad i \neq k. \end{cases}$$

$A \Rightarrow B$ means that from statement A the statement B follows; $A \Leftrightarrow B$ says that the statements A and B are equivalent (each follows from the other). The end of a proof is usually marked by ■.

Sets

$\varnothing$ is the empty set. $A \subset B$ means that A is a subset of B (where $A = B$ is permitted). For the formation of unions and intersections the signs $\cup$ and $\cap$ will be used, respectively. The *difference set* $E \setminus M$ is the set of all elements of E which do not belong to M; if M is a subset of E then $E \setminus M$ is also called the *complement* of M in E.

If the set M consists of all elements of a set E which possess a certain property P, we write $M = \{x \in E : x \text{ possesses property } P\}$. Examples (where at the same time certain symbols are defined): $[\alpha, \beta] := \{\xi \in \mathbf{R}: \alpha \leqq \xi \leqq \beta\}$ is the closed, $(\alpha, \beta) := \{\xi \in \mathbf{R}: \alpha < \xi < \beta\}$ is the open interval of the real line with endpoints α, β.

Maps

Let E, F be nonempty sets. A *map* f from E into F associates with each $x \in E$ one and only one element $y \in F$, which is also denoted $f(x)$ and is called the *image* of x (with respect to f). E is the *set of definition*, F the *target set* of f. In order to exhibit clearly the three components of a map (rule of correspondence f, set of definition E, target set F), we write

$$f: E \to F \quad \text{or} \quad f: \begin{cases} E \to F \\ x \mapsto f(x) \end{cases}$$

($x \mapsto f(x)$ means that with the element x one associates the image $f(x)$). The notation $f: x \mapsto f(x)$ or even simpler $x \mapsto f(x)$ is also used; then the set of definition and the target set must be given separately, if they are not evident. At times it will be handy—and harmless—to infringe on these notational conventions. Thus, for instance, we shall speak of the function $\sin x$ (instead of $x \mapsto \sin x$), of the polynomial x^2 (instead of $x \mapsto x^2$), and of the kernel $k(s, t)$ (instead of $(s, t) \mapsto k(s, t)$) of an integral equation. A *self-map* of E is a map from E into E. The *identity map* i_E of E is the map $x \mapsto x$ $(x \in E)$.

Two maps $f_1: E_1 \to F_1, f_2: E_2 \to F_2$ are said to be *equal* if $E_1 = E_2, F_1 = F_2$ and $f_1(x) = f_2(x)$ for all $x \in E_1$.

Let $f: E \to F$ be given. For $A \subset E, B \subset F$ the set $f(A) := \{f(x) \in F: x \in A\}$ is the *image* of A, $f^{-1}(B) := \{x \in E: f(x) \in B\}$ is the *preimage* of B. f is said to be *surjective* if $f(E) = F$, *injective* if $f(x) = f(y)$ implies $x = y$, and *bijective* if it is both surjective and injective. The locution 'f maps E onto F' means that f is surjective. A *family* $(a_\iota: \iota \in J)$ is only another name and writing for the map $\iota \mapsto a_\iota$ from a *set J of indices* into a set A. When $J = \mathbf{N}$ one rather speaks of a *sequence* than of a family.

A sequence $a_1, a_2, \ldots$ of elements of A will be denoted briefly by (a_n) or $(a_n) \subset A$ and occasionally also by $(a_1, a_2, \ldots)$.

The word *function* (and also *functional*) will in general be used only for maps from a set E into the field of scalars $\mathbf{K}$ (scalar-valued or $\mathbf{K}$-valued maps).

If the maps $f: E \to F$, $g: F \to G$ are given (observe that the target set of f is the set of definition of g), then their *composition* (*product*) is the map $g \circ f$ from E into G which associates with each $x \in E$ the image $(g \circ f)(x) := g(f(x))$ in G. For a self-map f of E, the *iterates* (*powers*) f^n are defined by $f^0 := i_E$, $f^n := f \circ f^{n-1}$ $(n = 1, 2, \ldots)$.

Every injective map $f: E \to F$ has an *inverse* map

$$f^{-1}: \begin{cases} f(E) \to E \\ f(x) \mapsto x. \end{cases}$$

One has $f^{-1} \circ f = i_E, f \circ f^{-1} = i_{f(E)}$.

$f: E \to F$ is then and only then bijective, if there exist maps $g: F \to E$, $h: F \to E$ with $g \circ f = i_E$, $f \circ h = i_F$. In this case $g = h = f^{-1}$.

The following rules will be used frequently for the study of maps $f: E \to F$ (let A, A_ι be subsets of E, while B, B_ι are subsets of F; $f^{-1}(B)$ is the above defined preimage, where f is not assumed to be injective):

$$A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2);$$

$$f\Big(\bigcap_{\iota \in J} A_\iota\Big) \subset \bigcap_{\iota \in J} f(A_\iota), \qquad f\Big(\bigcup_{\iota \in J} A_\iota\Big) = \bigcup_{\iota \in J} f(A_\iota);$$

$$B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2);$$

$$f^{-1}\Big(\bigcup_{\iota \in J} B_\iota\Big) = \bigcup_{\iota \in J} f^{-1}(B_\iota), \qquad f^{-1}\Big(\bigcap_{\iota \in J} B_\iota\Big) = \bigcap_{\iota \in J} f^{-1}(B_\iota);$$

$$f^{-1}(F \setminus B) = E \setminus f^{-1}(B);$$

$$f^{-1}(f(A)) \supset A; f \text{ injective} \Leftrightarrow f^{-1}(f(A)) = A \text{ for every } A \subset E;$$

$$f(f^{-1}(B)) \subset B; f \text{ surjective} \Leftrightarrow f(f^{-1}(B)) = B \text{ for every } B \subset F.$$

If the map $g: F \to G$ is also given, then

$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$$

for every $C \subset G$.

The *cartesian product* of a family $(E_\iota; \iota \in J)$ of non-empty sets E_ι is the set of all families $(x_\iota \in E_\iota: \iota \in J)$, i.e., the set of all maps $\iota \mapsto x_\iota \in E_\iota$ defined on J. It will be denoted $\prod_{\iota \in J} E_\iota$, and in the case of a finite or countable index set also by $E_1 \times \cdots \times E_n$ or $E_1 \times E_2 \times \ldots$, respectively. $E_1 \times \cdots \times E_n$ is the set of all n-tuples $(x_1, \ldots, x_n)$, $x_k \in E_k$ for $k = 1, \ldots, n$, while $E_1 \times E_2 \times \ldots$ is the collection of all sequences $(x_1, x_2, \ldots)$ with $x_k \in E_k$ for $k \in \mathbb{N}$. There is no risk of confusing a couple $(x_1, x_2) \in E_1 \times E_2$ with an open interval.

For a given $x := (x_\iota: \iota \in J) \in \prod_{\iota \in J} E_\iota$ the element x_ι is called the *component* of x in E_ι; the maps $\pi_\iota: (x_\iota: \iota \in J) \mapsto x_\iota$ are called the *projections* or *projectors* onto the components.

Rules of complementation

Let $(A_\iota: \iota \in J)$ be a family of subsets of E and $M' := E \setminus M$ the complement of $M \subset E$ in E. Then

$$\Big(\bigcup_{\iota \in J} A_\iota\Big)' = \bigcap_{\iota \in J} A'_\iota, \qquad \Big(\bigcap_{\iota \in J} A_\iota\Big)' = \bigcup_{\iota \in J} A'_\iota.$$

Zorn's lemma

For certain pairs x, y of elements of a set $\mathfrak{M} \neq \varnothing$ let a relation '$x \prec y$' be defined, which satisfies the following axioms:

1. $x \prec x$ for every $x \in \mathfrak{M}$.
2. If $x \prec y$ and $y \prec x$, then $x = y$.
3. If $x \prec y$ and $y \prec z$, then $x \prec z$.

Such a relation is called an *order* on $\mathfrak{M}$ and $\mathfrak{M}$ itself is said to be an *ordered set*. An order is called *total*, and $\mathfrak{M}$ a *totally ordered set*, if two elements x, y of $\mathfrak{M}$ are always *comparable*, i.e., if either $x \prec y$ or $y \prec x$ holds. Every subset of $\mathfrak{M}$ becomes through the order on $\mathfrak{M}$ an ordered, possibly even a totally ordered set. $y \in \mathfrak{M}$ is called an *upper bound* for $\mathfrak{N} \subset \mathfrak{M}$ if $x \prec y$ for every $x \in \mathfrak{N}$. $z \in \mathfrak{M}$ is a *maximal element* if $z \prec x$ holds only for $x = z$. Zorn's lemma is as follows:

If every totally ordered *subset of an ordered set $\mathfrak{M}$ has an upper bound in $\mathfrak{M}$, then there exists in $\mathfrak{M}$ at least one* maximal *element.*

Inequalities

The quantities α_k, β_k, $f(x)$, $g(x)$ which occur in what follows are complex numbers. The sums are finite or infinite; in the latter case it will be supposed that every series, which is on the right hand side of an inequality, converges. In the integral inequalities it is enough to assume for our purposes that the integrands are continuous functions on a finite interval of the real line. Proofs can be found in [180].

Hölder's inequalities: If $p > 1$ and $1/p + 1/q = 1$, then

$$\sum |\alpha_k \beta_k| \leqq \left(\sum |\alpha_k|^p\right)^{1/p} \left(\sum |\beta_k|^q\right)^{1/q},$$

$$\int_a^b |f(x)g(x)|\,dx \leqq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \left(\int_a^b |g(x)|^q\,dx\right)^{1/q};$$

for $p = q = 2$ one obtains the

Cauchy-Schwarz inequalities:

$$\sum |\alpha_k \beta_k| \leqq \left(\sum |\alpha_k|^2\right)^{1/2} \left(\sum |\beta_k|^2\right)^{1/2},$$

$$\int_a^b |f(x)g(x)|\,dx \leqq \left(\int_a^b |f(x)|^2\,dx\right)^{1/2} \left(\int_a^b |g(x)|^2\,dx\right)^{1/2}.$$

Minkowski's inequalities: For $p \geqq 1$ one has

$$\left(\sum |\alpha_k + \beta_k|^p\right)^{1/p} \leqq \left(\sum |\alpha_k|^p\right)^{1/p} + \left(\sum |\beta_k|^p\right)^{1/p},$$

$$\left(\int_a^b |f(x) + g(x)|^p\,dx\right)^{1/p} \leqq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}.$$

For $p > 1$ equality holds in the Minkowski inequalities if and only if one of the sequences (α_k), (β_k) or one of the functions f, g is a non-negative multiple of the other.

Labeling of results, references

To indicate the significance of the results, we use the hierarchy 'Lemma, Proposition, Theorem, Principle'. The statements within each class will be numbered consecutively in each section, thus Lemma 27.1 is the first lemma in §27, Proposition 27.1 the first proposition in §27, and Theorem 27.1 the first theorem in §27. The examples are numbered correspondingly. The exercises are at the end of each section; cross-references like '§16 Exercise 6' or 'Exercise 6 in §16' need no explanation. The sections (paragraphs) are numbered consecutively (§1 to §110). Square brackets refer to the bibliography. For the easier orientation of the reader it is divided into several sections, which are of course not completely independent of each other. We call the reader's attention especially to the section 'expository articles'; in them he will be made familiar easily and thoroughly with historical development, the essential problems, the fundamental notions and methods of certain fields of functional analysis; most of these articles also contain a very detailed bibliography.

Exercises

The exercises form an essential part of this book. They serve to get practice in the knowledge and methods of the main text (and give thereby also an opportunity for the reader to check whether he has understood it), they prepare for subsequent developments and communicate further interesting facts from functional analysis. Some exercises will be needed in the course of the main text, they are marked with a star in front of their number (e.g., *5). The reader is earnestly urged to work the exercises, the starred ones as well as the unstarred ones; the very hasty user of this book should at least glance through them. Those unstarred exercises which contain particularly interesting facts from functional analysis, which are not treated in the main text, are indicated by a plus sign in front of the number (e.g., $^{+}2$).

Guide

Basic are Chapters I–III and sections 28, 31–33. The following division according to subjects does not contain them any more. The division is to be understood only as an orientation; it does not show the existing logical interdependence:

(A) Geometry of spaces:
  1. Normed spaces: Chapter VIII (§41 must be read before §61)
  2. Inner product spaces: §§57–59, Chapter IX
  3. Topological vector spaces: Chapters XI–XIII

(B) Banach algebras: §§44–48, Chapter XIV
(C) Operators on normed spaces: §§29, 36, 41, 44–50, 53, 54, 97
(D) Operators on inner product spaces: §67, Chapter IX
(E) Fredholm operators: Chapter IV, §§37, 39, 51, 89
(F) Compact and Riesz operators: §30, Chapter VI, §§52, 72, 75.

I

Banach's fixed point theorem

§1. Metric spaces

One of the basic concepts of classical analysis is the concept of *convergence* of a sequence of numbers. This in turn is based on the concept of *distance*: in fact the convergence of the sequence (x_k) to x means that *the distance* $|x_k - x|$ *of the k-th term* x_k *from the limit x will be arbitrarily small when k increases beyond all bounds.* A corresponding definition is given for sequences of elements of $\mathbf{K}^n$, where the distance $d(x, y)$ between $x = (\xi_1, \ldots, \xi_n)$ and $y = (\eta_1, \ldots, \eta_n)$ is for instance defined by

$$d(x, y) := \left(\sum_{\nu=1}^{n} |\xi_\nu - \eta_\nu|^2 \right)^{1/2}. \tag{1.1}$$

If one wants to build a theory of convergence, which is valid for sequences of numbers as well as for sequences of vectors, and which eventually disregards completely the nature of the terms of the sequences, then one cannot use definitions of a distance like (1.1) (which make sense only for concretely given objects), but rather one will have to work with some properties, which intuitively every reasonable concept of distance must have. Such properties are for instance the following, where we call 'points' the objects between which the distances are defined: The distance of a point from itself and only from itself is 0, the distance of a point x from a point y is exactly as large as conversely the distance of the point y from the point x (symmetry property of distance), and finally a 'deviation property': If one does not go from the point x directly to the point y but first to the point z and then from there to x, then one has not made a shortcut, possibly one covered a larger distance. We need now only to express these properties precisely in the language of mathematical formulas to obtain the fundamental concepts of a metric and of a metric space: