KNOT PROJECTIONS

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To my teachers, family,
and friends.

Preface

If you see something, do you understand what it is?

You may be fortunate enough to possibly be able to grab it with your hand; however, simply holding something does not mean you understand it well. For example, what would you think about Figure 0.1. Figure 0.1 shows an example of a *knot diagram*.

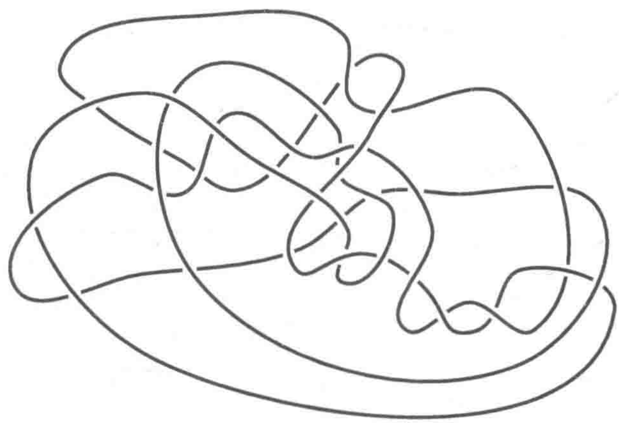

FIGURE 0.1 Knot diagram

In fact, there are many such things in the world that we can see and touch but may not understand what they are.* For some mathematicians, it is elementary to determine whether two representations corresponding to the same knot (= same closed string object), which are shown in Figure 0.2; Figure 0.2 shows two representations of the most simple non-trivial knot.

However, it would *not* be elementary to determine the same for the next pair as shown in Figure 0.3. In fact, this pair is called a version of the *Perko pair*† representing the same knot.

Historically, Perko found duplication in a knot table from the nineteenth century, which implies that no one pointed out the duplication for about 75

*The knot diagram in Figure 0.1 represents a trivial knot, which was constructed by referring to [1] and "The Culprit Undone" in [2, 4].

†"Perko pair" is a famous term among specialists. However, the author found several versions of the Perko pair and noticed that it may not represent a unique pair.

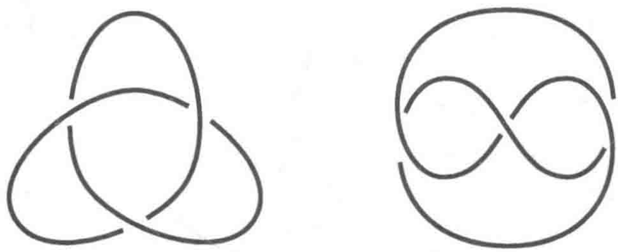

FIGURE 0.2 Two representations of the same knot

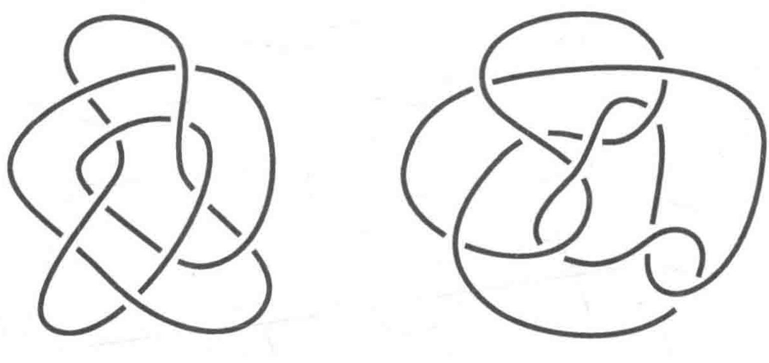

FIGURE 0.3 Perko pair

years. Hence, we can say that, in general, it is difficult to determine whether two knot diagrams truly represent two different knots or actually represent the same knot. Perko obtained a deformation (called an isotopy) from the left figure to the right figure, as shown in Figure 0.4.

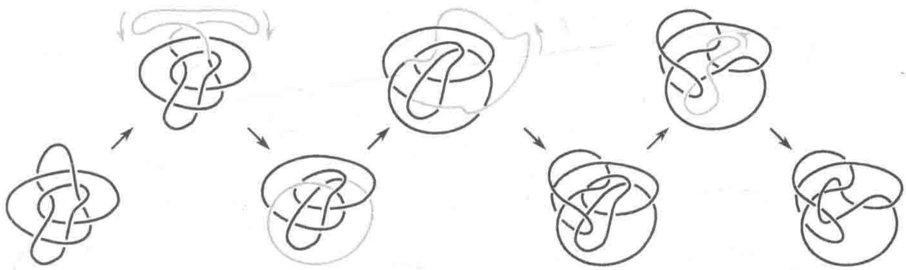

FIGURE 0.4 Isotopy from the left to the right figure

If we omit every over/under-information of a crossing of a knot diagram, the result is called a *knot projection*. Is it still difficult to determine what a knot projection is?

In the 1920s, Reidemeister obtained a result that showed that RI, RII, and RIII are sufficient to describe a deformation of any knot projection into a simple closed curve, where RI, RII, and RIII are local replacements between two knot projections as shown in Figure 0.5. Traditionally, in mathematics, such a deformation is treated as a notion of *homotopy*.

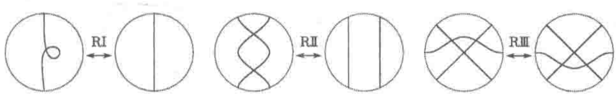

FIGURE 0.5 Reidemeister moves RI, RII, and RIII

In 2001, Östlund formulated a question as follows:
Östlund's question: Let RI and RIII denote the local replacements between two knot projections as shown in Figure 0.5. For every knot projection P, are RI and RIII sufficient to describe a deformation of P into a simple closed curve?

In 2014, Hagge and Yazinski were the first to obtain an example to answer to this question, which is shown in Figure 0.6.

However, a function on the set of the knot projections that can be used to easily detect which knot projection is related to a simple closed curve by a finite sequence generated by RI and RIII has not been found. We still do not have sufficient information about homotopies generated by RI and RIII for knot projections and more research on knot projections is needed.

The objective of this book to address the classification problem of knot projections and discuss related topics. This monograph consists of 11 chapters.

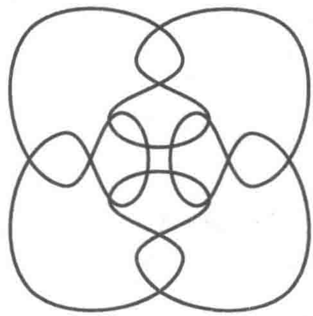

FIGURE 0.6 Example obtained by Hagge and Yazinski

Chapter 1 provides the definitions of knots and knot projections; in Chapter 2, we learn why we consider Reidemeister moves from a result obtained in the 1920s (Reidemeister's Theorem). In Chapter 3, we describe a useful result obtained by Whitney (1930s), i.e., two knot projections with the same rotation number on a plane if and only if they are related by a finite sequence generated by RII, RIII, and a plane isotopy. In Chapter 4, we reach the end of the 20th century, where Khovanov's Theorem (1997) and its proof are described; this theorem can be used to obtain the classification of knot projections under homotopy generated by RI, RII, and sphere isotopy.

After obtaining the basic results based on the studies of Reidemeister, Whitney, and Khovanov, we discuss most recent results, i.e., from 2013–2015. We consider classifications using refined Reidemeister moves from the latter part of Chapter 4 and Chapter 5. We decompose RII into strong RII and weak RII; similarly, we decompose RIII into strong RIII and weak RIII. This decomposition is performed to not only provide a more detailed classification from a general one but also to open a new viewpoint of this mathematical area.

Historically, Arnold discussed classification theory using the singularity theory in mathematics to classify plane curves (= knot projections on a plane) under strong RII and RIII, weak RII and RIII, or RII, i.e., without RI. The structure of a differential that is used to obtain a good classification of knot projection leads to the notion of *Legendrian knots*; however, RI cannot be used. By contrast, following Arnold's theory, many mathematicians have obtained excellent results, which implies a good understanding of a classification with respect to the equivalence relation generated by strong RII and weak RIII. Many of these works provide ideas to study knot projections further.

Thus, finally, we study the classification theory of knot projections by considering a pair consisting of RI and other Reidemeister moves, denoted by $(\mathrm{RI}, \cdot)$, as shown in the following list. Each theory should be constructed depending on the situation of each case because there is difficulty when considering RI as mentioned above. In these chapters, we introduce ideas to obtain new invariants of knot projections. Here, an invariant of knot projections is a function on the set of the knot projections. Invariants are useful to determine which two knot projections are not equivalent. In other words, for a case where there is a situation such that there exist two different knot projections with the same invariant values, the classification problem is still open.

- (RI, strong RII) or (RI, weak RII): the latter part of Chapter 4,
- (RI, strong RIII): Chapter 5,
- (RI, strong RIII) or (RI, weak RIII): Chapter 6,
- (RI, RIII): Chapter 7,
- (RI, strong RIII) revisited: Chapter 8,
- (RI, weak RII, weak RIII): Chapter 10

Chapter 9 discusses topics related to (RI, strong RII), i.e., unavoidable sets. In topological studies, "unavoidable sets" is a famous term with respect to the four-color theorem. In Chapter 9, unavoidable sets for knot projections as described by Shimizu are introduced. Several definitions of reductivity related to unavoidable sets and characterizations are introduced, based on the works in [6] and [3].

In the last chapter, Chapter 11, we appreciate Viro's remarkable work, i.e., quantization of Arnold invariants of knot projections on a plane; his work obtains an infinite family of invariants under (weak RII, weak RIII). This is one of the most important goals achieved by Arnold's theory for plane curves. However, there has been no such quantum theory for knot projections with respect to (RI, $\cdot$) or a unified theory of invariants yet. Therefore, this chapter is a part of the epilogue of this book.

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Noboru Ito
Tokyo, February 2016

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Contents

CHAPTER 1

Knots, knot diagrams, and knot projections

CONTENTS

Knot projections are knots in a three-dimensional space as closed string objects. Knots are not only common in the field of mathematics but also in some areas of theoretical physics and biology. In this chapter, we define knots, knot diagrams, and knot projections.

1.1 DEFINITION OF KNOTS FOR HIGH SCHOOL STUDENTS

A *knot* is the union of a finite number of straight line segments with no boundaries in $\mathbb{R}^3$ such that each endpoint joins exactly one of the remaining endpoints. Here, $\mathbb{R}^3$ is defined as $\{(x, y, z) \mid x, y, z \text{ are real numbers}\}$. We can use a smooth plane curve, called a *knot diagram*, to represent a knot (see Section 1.4), which is shown in Figure 1.1 (left). A straight line segment is called an *edge* and an endpoint is called a *vertex*.

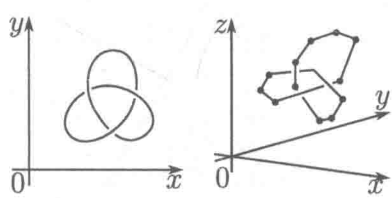

FIGURE 1.1 A knot (right) and its knot diagram (left)