José Ferreirós

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Foreword

The philosophy of mathematics has experienced a renewal in recent years due to a more open and interdisciplinary way of asking and answering questions. Traditional philosophical concerns about the nature of mathematical objects and the epistemology of mathematics are combined and fructified with the study of a wide variety of issues about the way mathematics is done, evaluated, and applied—and in connection therewith, about historical episodes or traditions, educational problems, cognitive questions. The outcome is a broad outward-looking approach to the philosophy of mathematics, which engages with mathematics in practice.[1] This book is an outcome of such initiatives.

Purists may perhaps feel that this is not philosophy anymore; but, in my opinion, such judgments come from an unduly narrow understanding of the philosophical enterprise. In any event, it is our hope that a broad, open-minded approach to the study of mathematical practice can only act for the good. The study of mathematical knowledge and how it is produced is an important topic, and it is certainly desirable that the work of philosophers be of interest to mathematicians, mathematical educationalists, and scientists who are mathematics users.

In spite of the rather new orientation, there is a sense in which this book deals with traditional topics in the philosophy of mathematics. There are many interesting questions about the practice of mathematics (explanation, fruitfulness, computers, cognitive roots, and more) that I don't discuss carefully enough, and in some cases not at all.[2] This is not because of a lack of interest in them, but simply because I believed I had something to say about more traditional problems—questions about mathematical knowledge and its objectivity—from the standpoint of

[1] Cf. the description of the aims and goals of the APMP, which I have merely made mine in these sentences: http:// institucional.us.es/apmp/index_about.htm.

[2] A very good introduction to this area is the collective volume Mancosu (2008).

an approach that emphasizes the web of intertwined knowledge and practices, and the role of agents.

The main topics discussed have to do with core mathematical theories, in fact, with those structures that practicing mathematicians tend to regard as reflecting categorical models. (An axiom system is said to be *categorical* when all its models are isomorphic; traditionally, the theory of real numbers has been regarded as categorical, although this is more a theoretical desideratum than a fact of mathematical logic; the same applies of course to the natural-number structure.) Not much will be said about the natural numbers, since the topic is abundantly treated in other places and I wanted to concentrate on something different. There is a sustained argument about how the mathematical tradition evolved from Euclidean geometry to the real numbers, and from them to set theory.

I know some readers may be disappointed that I don't concentrate on very advanced material (e.g., on the worlds of contemporary mathematics), but I do believe the philosopher still has to deal adequately with more basic stuff.

After Chapter 1, which is introductory and written as a manifesto, Chapters 2–4 form a philosophical introduction to the standpoint from which the whole book is written. Chapter 2 can be read as a general introduction to the current trend of studies of mathematical practice, focusing above all on historical and philosophical work; it proposes a preliminary explanation of the notion of mathematical practice. Chapter 3 goes to the heart of the matter by laying out some crucial ingredients for the historically oriented, agent-based philosophy of mathematics that I develop (a web of practices based always on agents; symbolic frameworks; theoretical frameworks). Chapter 4 presents a thesis about what I consider to be a crucial *complementarity* of elements in the practice of mathematics, and in the configuration of mathematical knowledge: both conceptual and formalistic ingredients enter the game, both necessary, and neither reducible to the other.

As should be obvious from this brief overlook, the first four chapters are heavily philosophical in orientation. But the reader should feel free to employ this book in several different ways. Readers interested more directly in the mathematical material might go to Chapter 5 directly, perhaps after having read the "manifesto" in Chapter 1. In fact, some readers might even have the impression that the philosophical content in the first four chapters is not used very much in the second half of the book. While it is true that I have not paused to make explicit how the analysis of mathematical knowledge in the second part depends on the philosophical perspective presented in the first part (mainly because this would have considerably lengthened the work), nonetheless that perspective is presupposed. Readers versed in philosophy would certainly find my discussion utterly incomplete if I had left out the material in Chapters 2–4.

Chapter 5 considers the ancient tradition of geometrical proof in the light of recent studies by Manders and others. This in my opinion is highly important, since our *leitfaden* throughout the book is the idea of the continuum, and this originates in geometry (well, to be more precise, in notions of space, time, and movement). Also, the analysis of old geometric practices is a paradigm of the philosophy of mathematical practice, showing most clearly how this is different from traditional philosophy and how it complements and expands twentieth-century conceptions of proof. And, not least, mathematicians who approach the matter with an open mind (both researchers and students) will be intrigued by the challenge that Euclid poses to their basic assumptions about math.

Chapter 6 presents a crucial tenet of this work, the idea that advanced mathematics is based on hypotheses—that far from being a priori, it is based on *hypothetical* assumptions. The fact that an axiom is well established and generally accepted does not detract from its being, as I state, hypothetical in nature.[3] I do not claim originality here, since one can find related ideas in a long and important tradition of reflection on mathematical knowledge (including Riemann, Peirce, Poincaré, Weyl, Quine, Putnam, and others). What is original in my study is the way I elaborate on the idea and its implications: the devil is in the details. As that standpoint—the way I present it—depends on a distinction between elementary and advanced math, Chapter 7 studies some aspects of the paradigm of elementary math, basic arithmetic. I elaborate on the idea

[3] To be a bit precise, examples I have in mind include the Axiom of Completeness for the real number system, the Axioms of Infinity and Powerset in set theory, and the Axiom of Choice in category-theoretic foundations.

that we have *certainty* in our basic arithmetic knowledge. This is unlike what happens in advanced mathematical topics, which means (or so I claim) almost all of mathematics, beginning with plane geometry and the theory of real numbers (studied in Chapter 8).

Perhaps the most interesting element in this work will be the argument I put forward for reconciling the hypothetical conception of mathematics with the traditional idea of the objectivity of mathematical knowledge. This is the purpose of Chapter 9, to which I direct the reader's attention. The basic notion is that, because new hypotheses are embedded in the web of mathematical practices, they become systematically linked with previous strata of mathematical knowledge, and this forces upon us agents (e.g., research mathematicians or students of math) certain results, be they principles or conclusions. Mathematical knowledge is neither invention nor discovery: in advanced math we rather find something like invention-*cum*-discovery,[4] which shows that the simple dichotomy (invention vs. discovery) is quite useless for the analysis of scientific knowledge.

Finally, Chapter 10 is a preliminary look into one of the most intriguing questions that a philosophy of mathematics in practice must, sooner or later, confront: how *understanding* of math is obtained.

Let me insist on the idea that the reader has several options. Mathematically inclined readers might want to concentrate on the second half of this work, reading sections of the early chapters along the way, if and as they feel the need to have a stronger basis for the philosophical aspects. Philosophically inclined readers will certainly want to start with the early chapters, but they are invited to judge the perspective presented in them by its fruits in the concrete analysis of mathematical material discussed in the last few chapters. This is not an academic book of the kind that tries to consider all the alternative viewpoints, arguing for or against each of them. It is rather an attempt to explore in depth the possibilities of one particular take on the issues; by their fruits ye shall know them.

As for requirements, neither advanced knowledge of mathematics nor of philosophy is presupposed—yet it's true that some things have

[4] In contrast, in the case of elementary math we essentially discover traits of the objective world (a world to which agents belong).

been presupposed. An effort has been made to write clearly, yet the book requires a certain level of maturity; as Frege suggested, the philosophy of mathematics is in a difficult position, since many mathematicians will regard it as too philosophical, and many philosophers will find it too mathematical. On the side of mathematics, it is expected that the reader will have studied a rigorous definition of the real numbers, but a careful reading of Dedekind's beautiful essay "Continuity and irrational numbers" should suffice.[5] Set theory is mentioned as an example in many passages, but the usual rather informal acquaintance with the theory should be enough for the reader. Acquaintance with the axiom system of Zermelo-Fraenkel, however, will certainly be an advantage.[6]

Other than that, there are passages in which I mention concepts such as the von Neumann ordinals, nonstandard models of arithmetic, or Goodstein's theorem. Since those are in no way essential to following the argument, I have taken the freedom to mention them without giving the details; this seems to be a viable option, particularly nowadays, when it is so easy to find basic information about any of those points in Wikipedia and other places.

On the side of philosophy, the concepts or principles required to understand the position advocated here have been given explicitly with sufficient clarity—I hope. Given that philosophy is, by nature, heavily argumentative, there are many passages devoted to answering possible worries or objections I anticipate. This may make things somewhat difficult for less expert readers, who may be unfamiliar with the points of view that I am taking into account. My counsel is simply to read through those passages without trying to fully grasp what is at stake. There's much to be gained from reading books of this kind, even when one does not attain full command of the discussion.

The ideas that I present in this work have long been under elaboration, and that makes it very difficult for me to acknowledge properly

[5] For a more advanced work, I can recommend the recent Stillwell 2013 (much of whose content deals with material that goes beyond what is treated here).

[6] See, e.g., Thomas Jech's entry on "Set Theory" in the *Stanford Encyclopedia of Philosophy*, edited by E. N. Zalta, and its supplement on Zermelo-Fraenkel Set Theory," http://plato.stanford.edu/archives/fall2014/entries/set-theory/ and http://plato.stanford.edu/entries/set-theory/ZF.html.

everyone who has influenced my thinking. In fact, I began to reflect seriously on this topic during my undergraduate studies in philosophy, decades ago. Thus, my readings in philosophy (Descartes, Berkeley, Leibniz, Kant, Russell, Wittgenstein, Piaget, Quine, Kuhn, Kitcher, etc.) have always been shaped by my interest in understanding mathematics. And then, my readings in mathematics (Gauss, Cantor, Dedekind, Riemann, Poincaré, Frege, Hilbert, Weyl, Gödel, etc.) have been shaped by my interest in philosophy. Of special importance to my way of thinking have been, as far as I can tell, at least three ingredients: the interdisciplinary orientation, which I have always followed (e.g., with excursions into psychology and cognitive science); the contrast between the areas of mathematics and logic on the one hand, and historical work on the other, a contrast that is full of methodological lessons; and, more generally, the contrasts between scientific and philosophical readings, e.g., between old theory of knowledge and issues in biology and cognitive science.

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Apologies to those whose names I may be overlooking. And thanks also to the personnel at Princeton University Press, in particular to Vickie Kearn and Jenny Wolkowicki, for their careful work transforming the manuscript into a well-edited book.

Despite everyone's good efforts, the flaws and peculiarities readers will find in this work, are a natural result of my stubborn ways of thinking.

The book is dedicated to Dolores, Inés, Lucía, and Juana, for all their love and care.

José Ferreirós
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On Knowledge and Practices

A Manifesto

Although the idea of emphasizing them is relatively new, and there is still some disunity concerning how to focus our analysis, mathematical practices are in the agenda of every practicing philosopher of mathematics today. Mathematical knowledge, on the other hand, has always figured prominently among the mysteries of philosophy.[1] Can we shed light on the latter by paying attention to the former? My answer is yes. I believe the time is ripe for an ambitious research project that targets mathematical knowledge in a novel way, operating from a practice-oriented standpoint.

Let us begin by placing this kind of enterprise within the context of the philosophy of mathematics. During the twentieth century, we have seen several different broad currents in this field, which, simplifying a great deal, can be reduced to three main types: *foundational* approaches (logicism, intuitionism, formalism, finitism, and predicativism), *analytic* approaches (focused on questions of ontology and epistemology), and the so-called "*maverick*" approaches (to use Kitcher's colorful terminology), which have typically been anti-foundational and focused on history, methodology, and patterns of change. Mixed approaches have, of course, been present throughout the century, although one can say that they remained relatively uninfluential; early examples are the work of Jean Cavaillès in France during the late 1930s and that of Paul Bernays in Germany and Switzerland from the 1930s on. But in the 1980s and

[1] A mystery that some twentieth-century philosophers (Wittgenstein prominent among them) tried to dispel like a simple fog, but quite unconvincingly. For reasons why mathematics is not a game of tautologies, nor a mere calculus of symbol transformations, see especially Chapters 4 and 6.