Bernt Øksendal

Stochastic Differential Equations

An Introduction with Applications

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Preface

These notes are based on a postgraduate course I gave on stochastic differential equations at Edinburgh University in the spring 1982. No previous knowledge about the subject was assumed, but the presentation is based on some background in measure theory.

There are several reasons why one should learn more about stochastic differential equations: They have a wide range of applications outside mathematics, there are many fruitful connections to other mathematical disciplines and the subject has a rapidly developing life of its own as a fascinating research field with many interesting unanswered questions.

Unfortunately most of the literature about stochastic differential equations seems to place so much emphasis on rigor and completeness that is scares many nonexperts away. These notes are an attempt to approach the subject from the nonexpert point of view: Not knowing anything (except rumours, maybe) about a subject to start with, what would I like to know first of all? My answer would be:

1) In what situations does the subject arise?
2) What are its essential features?
3) What are the applications and the connections to other fields?

I would not be so interested in the proof of the most general case, but rather in an easier proof of a special case, which may give just as much of the basic idea in the argument. And I would be willing to believe some basic results without proof (at first stage, anyway) in order to have time for some more basic applications.

These notes reflect this point of view. Such an approach enables us to reach the highlights of the theory quicker and easier. Thus it is hoped that notes may contribute to fill a gap in the existing literature. The course is meant to be an appetizer. If it succeeds in awaking further interest, the reader will have a large selection of excellent literature available for the study of the whole story. Some of this literature is listed at the back.

In the introduction we state 6 problems where stochastic differential equations play an essential role in the solution. In Chapter II we introduce the basic mathematical notions needed for the mathematical model of some of these problems, leading to the concept of Ito integrals in Chapter III. In Chapter IV we develop the stochastic calculus (the Ito formula) and in Chapter V we use this to solve some stochastic differential equations, including the first two problems in the introduction. In Chapter VI we present a solution of the linear filtering problem (of which problem 3 is an example), using the stochastic calculus. Problem 4 is the Dirichlet problem. Although this is purely deterministic we outline in Chapters VII and VIII how the introduction of an associated Ito diffusion (i.e. solution of a stochastic differential equation) leads to a simple, intuitive and useful stochastic solution, which is the cornerstone of stochastic potential theory. Problem 5 is (a discrete version of) an optimal stopping problem. In Chapter IX we represent the state of a game at time t by an Ito diffusion and solve the corresponding optimal stopping problem. The solution involves potential theoretic notions, such as the generalized harmonic extension provided by the solution of the Dirichlet problem in Chapter VIII. Problem 6 is a stochastic version of F. P. Ramsey's classical control problem from 1928. In chapter X we formulate the general stochastic control problem in terms of stochastic differential equations, and we apply the results of Chapters VII and VIII to show that the problem can be reduced to solving the (deterministic) Hamilton-Jacobi-Bellman equation. As an illustration we solve a problem about optimal portfolio selection.

After the course was first given in Edinburgh in 1982, revised and expanded versions were presented at Agder College, Kristiansand and University of Oslo. Every time about half of the audience have come from the applied section, the others being so-called "pure" mathematicians. This fruitful combination has created a broad variety of valuable comments, for which I am very grateful. I particularly wish to express my gratitude to K. K. Aase, L. Csink and A. M. Davie for many useful discussions.

I wish to thank the Science and Engineering Research Council, U.K. and Norges Almenvitenskapelige Forskningsråd (NAVF), Norway for their financial support. And I am greatly indebted to Ingrid Skram, Agder College and Inger Prestbakken, University of Oslo for their excellent typing – and their patience with the innumerable changes in the manuscript during these two years.

Oslo, June 1985 Bernt Øksendal

We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things.

Posted outside the mathematics reading room,
Tromsø University

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I. Introduction

To convince the reader that stochastic differential equations is an important subject let us mention some situations where such equations appear and can be used:

(A) <u>Stochastic analogs of classical difference and differential equations</u>

If we allow for some randomness in some of the coefficients of a difference or differential equation we often obtain a more realistic mathematical model of the situation.

<u>PROBLEM 1</u>. Consider the simple population growth model

(1.1) $$\frac{dN}{dt} = a(t)N(t), \quad N(0) = A$$

where $N(t)$ is the size of the population at time t, and $a(t)$ is the relative rate of growth at time t.
It might happen that $a(t)$ is not completely known, but subject to some random environmental effects, so that we have

$$a(t) = r(t) + \text{"noise"},$$

where we do not know the exact behaviour of the noise term, only its probability distribution.
How do we solve (1.1) in this case?

PROBLEM 2. The charge $Q(t)$ at time t at a fixed point in an electric circuit satisfies the differential equation

$$L\cdot Q''(t) + R\cdot Q'(t) + \frac{1}{C}\cdot Q(t) = F(t),\ Q(0) = Q_0,\ Q'(0) = I_0 \tag{1.2}$$

where L is inductance, R is resistance, C is capacitance and $F(t)$ the potential source at time t.

Again we may have a situation where some of the coefficients, say $F(t)$, are not deterministic but of the form

$$F(t) = G(t) + \text{"noise"}.$$

How do we solve (1.2) in this case?

(B) Filtering problems

It is clear that any solution of the stochastic equations in Problem 1 and 2 must involve some randomness, i.e. we can only hope to be able to say something about the probability distributions of the solutions.

PROBLEM 3. Suppose that we, in order to improve our knowledge about the solution, say of Example 2, perform observations

$$z(t_1), \ldots, z(t_n) \tag{1.3}$$

of Q at times $t_1, \ldots, t_n$. However, due to inaccuracies in our measurements we do not really measure $Q(t_k)$ but a disturbed version of it:

$$z(t_k) = Q(t_k) + \text{"noise"}. \tag{1.4}$$

So in this case the noise term comes from the error of measurement.

The problem is: What is the best estimate of $Q(t)$ satisfying (1.2), based on the observations (1.4), where $t \geq t_n$?

If $t > t_n$ this is called a prediction problem, if $t = t_n$ it is called a filtering problem.

In 1960 Kalman and in 1961 Kalman and Bucy proved what is now known as the Kalman-Bucy filter. Basically the filter gives a procedure for estimating the state of a system which satisfies a "noisy" linear differential equation, based on a series of "noisy" observations.

Almost immediately the discovery found applications in aerospace engineering (Ranger, Mariner, Apollo etc.) and it now has a broad range of applications.

Thus the Kalman-Bucy filter is an example of a recent mathematical discovery which has already proved to be useful - it is not just "potentially" useful.

It is also a counterexample to the assertion that "applied mathematics is bad mathematics" and to the assertion that "the only really useful mathematics is the elementary mathematics". For the Kalman-Bucy filter - as the whole subject of stochastic differential equations - involves advanced, interesting and first class mathematics.

(C) Stochastic approach to deterministic boundary value problems

PROBLEM 4. The most celebrated example is the stochastic solution of the Dirichlet problem:

Given a (reasonable) domain U in $\mathbb{R}^n$ and a continuous function f on the boundary of U, ∂U.
Find a function $\tilde{f}$ continuous on the closure $\bar{U}$ of U such that

(i) $\tilde{f} = f$ on ∂U

(ii) $\tilde{f}$ is harmonic in U, i.e.

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0 \quad \text{in } U .$$

In 1944 Kakutani proved that the solution could be expressed in terms of Brownian motion:
$\tilde{f}(x)$ is the expected value of f at the first exit point from U of the Brownian motion starting at $x \in U$.

It turned out that this was just the tip of an iceberg: For a large class of semielliptic 2^{nd} order partial differential equations the corresponding Dirichlet boundary value problem can be solved using stochastic processes which are solutions of associated stochastic differential equations.

(D) Optimal stopping

PROBLEM 5. A participant in a contest is given a sequence of questions. If he gives the right answer to a question, he gets a reward plus the option of proceeding to the next question or withdraw from the contest (with the money he has received so far). If he cannot answer a question correctly he looses all the money he has received and he is out of the contest. Suppose that for each question there is a probability p that he can answer the question. When is the best time for the contestant to stop the game? What is the expected gain when stopping at such an optimal time?

This is (a discrete version of) an optimal stopping problem. It turns out that the solution can be expressed in terms of the solution of a corresponding generalized Dirichlet problem mentioned above.

(E) Stochastic control

PROBLEM 6. A stochastic analog of the "How much should a nation save?"-problem of F.P. Ramsey from 1928 (see [21]) in economics is the following:

The basic economic quantities are

$K(t)$ = capital at time t

$L(t)$ = labour at time t

$P(t)$ = production rate at time t

$C(t)$ = consumption rate at time t

$U(C)\Delta t$ = the "utility" obtained by consuming goods at the consumption rate C during the time interval Δt.

Let us assume that the relation between $K(t)$, $L(t)$ and $P(t)$ is of the Cobb-Douglas form:

$$P(t) = AK(t)^{\alpha}L(t)^{\beta}, \tag{1.5}$$

where A, α, β are constants.

Further, assume that

$$\frac{dK}{dt} = P(t) - C(t) \tag{1.6}$$

and

(1.7) $$\frac{dL}{dt} = a(t) \cdot L(t) ,$$

where $a(t) = r(t) +$ "noise" is the rate of growth of the population (labour) (see Example 1.1).

Given a utility function U and a "bequest" function ψ, the problem is to determine at each time t the size of the consumption rate $C(t)$ which maximizes the expected value of the total utility up to a future time $T \leq \infty$:

(1.8) $$\max \{E[\int_0^T U(C(t))e^{-\rho t}dt] + \psi(K(T))\}$$

where ρ is a discounting factor.

II. Some Mathematical Preliminaries

Having stated the problems we would like to solve, we now proceed to find reasonable mathematical notions corresponding to the quantities mentioned and mathematical models for the problems. In short, here is a first list of the notions that need a mathematical interpretation:

(1) A random quantity

(2) Independence

(3) Parametrized (discrete or continuous) families of random quantities

(4) What is meant by a "best" estimate in the filtering problem (Example 3)?

(5) What is meant by an estimate "based on" some observations (Example 3)?

(6) What is the mathematical interpretation of the "noise" terms?

(7) What is the mathematical interpretation of the stochastic differential equations?

In this chapter we will discuss (1) - (3) briefly. In the next chapter (III) we will consider (6), which leads to the notion of an Ito stochastic integral (7).
In chapters IV, V we consider the solution of stochastic differential equations and then return to a solution of Example 1.
In chapter VI we consider (4) and (5) and sketch the Kalman-Bucy solution to the linear filtering problem.
In chapter VII we investigate further the properties of a solution of a stochastic differential equation.
Then in chapters VIII, IX and X this is applied to solve the

generalized Dirichlet problem, optimal stopping problems and stochastic control problems, respectively.

The mathematical model for a random quantity is a random variable:

DEFINITION 2.1. A random variable is a $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$, where $(\Omega, \mathcal{F}, P)$ is a probability space. ($\mathcal{F}$ is a σ-algebra of subsets of Ω, P is a probability measure on Ω, assigning values in $[0,1]$ to each member of $\mathcal{F}$, B Borel set in $\mathbb{R}^n \Rightarrow X^{-1}(B) \in \mathcal{F}$.)

Every random variable induces a measure μ_X on $\mathbb{R}^n$, defined by

$$\mu_X(B) = P(X^{-1}(B)) .$$

μ_X is called the distribution of X.

The mathematical model for independence is the following:

DEFINITION 2.2. Two subsets $A, B \in \mathcal{F}$ are called independent if

$$P(A \cap B) = P(A) \cdot P(B) .$$

A collection $\mathcal{A} = \{\mathcal{O}_i \, ; \, i \in I\}$ of families $\mathcal{O}_i$ of measurable sets is/are independent if

$$P(A_{i_1} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \ldots P(A_{i_k})$$

for all choices of $A_{i_1} \in \mathcal{O}_{i_1}, \ldots, A_{i_k} \in \mathcal{O}_{i_k}$.

The σ-algebra $\mathcal{O}_X$ induced by a random variable X is

$$\mathcal{O}_X = \{X^{-1}(B) \, ; \, B \in \mathcal{B}\} ,$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$.

A collection of random variables $\{X_i \, ; \, i \in I\}$ are independent if the collection of induced σ-algebras $\mathcal{O}_{X_i}$ are independent.