



# *Matrix Inequalities for Iterative Systems*

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Iterative Systems*



*To my parents*  
*Brigitte and Klaus Täubig*



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# Acknowledgments

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*It seems to me shallow and arrogant for any man in these times to claim he is completely self-made, that he owes all his success to his own unaided efforts. Many hands and hearts and minds generally contribute to anyone's notable achievements.*

Walt Disney

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# Preface

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Our interest for the topic was attracted by a conjecture that was brought up by our colleague Sven Kosub. While we did not make much progress in proving the conjecture at the beginning (because it was wrong), it turned out later, that one of the first ideas for proving a certain special case was still at the heart of the more general results that were obtained only recently by Jeremias Weihmann and the author of this book.

This book is based on the author's habilitation thesis [Täu15a]. The main goal of the habilitation process is to develop and prove the ability to teach. As such, the aim of the book is to present the topic in such a way that the results and their underlying methods of proof can be used by the reader in the easiest possible way. To this end, our work tries to unify the various inequalities for the number of walks in graphs and for the sum of entries of matrix powers in a generalized form. These generalizations reveal the fundamental principles underlying the different results.

To the best of our knowledge all the results claimed to be new, in this work, have never appeared before. During the whole time of working on this subject, we found many examples of results that were (re)discovered more than just once. But in view of the large list of references, we can probably claim that we did a thorough job for finding most of the relevant literature. In any case, the focus of this book is not only on the collection, but also on the systematization of the vast number of results. We have generalized and unified the related results, and put them into a common frame and discussed their relations. In conclusion, we hope to contribute to a deeper understanding of the origin of the different inequalities for matrix powers and the number of walks in graphs. Our main goal is thus a clear presentation of the common principles underlying the tremendous number of different results.

Almost all of the proofs in this book are elementary. While it is normal to be satisfied with an arbitrary complex proof for a certain statement, it must be emphasized that elementary proofs (even for known results) have an additional value because they are instructive and convincing.

Most of the results in this book have been published in the following articles and reports: [HKM+11; HKM+12; Täu12; TW12; TWK+13; TW14; Täu14; Täu15b].



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# Symbol Description

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$A$	matrix with entries $a_{ij}$ , usually square of size $n \times n$	$G$	graph, usually with vertex set $V$ and edge set $E$
$A[X, Y]$	submatrix of $A$ induced by row index set $X$ and column index set $Y$	$V$	vertex set of a graph
$A[X]$	principal submatrix of $A$ induced by row and column index set $X$	$E$	edge set of a graph
$A^T$	transpose of $A$	$d_{\text{in}}(v)$	in-degree of vertex $v$
$A^*$	conjugate transpose of $A$	$d_{\text{out}}(v)$	out-degree of vertex $v$
$a_{ij}$	matrix entry that resides in row $i$ and column $j$	$\Delta$	maximum degree
$I_n$	$n \times n$ identity matrix	$\rho(G)$	edge density of the graph $G$
$\text{sum}(A)$	sum of all entries of $A$	$w_k(x, y)$	number of walks of length $k$ starting at vertex $x$ and ending at vertex $y$
$r_i(A), r_i$	row sum of the entries in row $i$ of the given matrix $A$	$s_k(x)$	number of walks of length $k$ starting at vertex $x$
$c_j(A), c_j$	column sum of the entries in column $j$ of the given matrix $A$	$e_k(x)$	number of walks of length $k$ ending at vertex $x$
$\text{tr}(A)$	trace of matrix $A$ (sum of main diagonal entries)	$w_k(x)$	replacement for $s_k(x)$ and $e_k(x)$ in undirected graphs
$\rho(A)$	spectral radius of $A$	$w_k$	total number of walks of length $k$ in a given graph
$\langle x, y \rangle$	(standard) inner product of vectors $x$ and $y$	$cl_k$	total number of closed walks of length $k$ in a given graph
$\ x\ $	(Euclidean) length of vector $x$	$cl_k(x)$	number of closed walks of length $k$ starting at vertex $x$
$\mathbf{1}_n$	$n$ -dimensional all-ones vector	$v_k$	total number of nonreturning walks of length $k$ in a given graph
$\chi(S)$	characteristic vector of set $S$		



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