Lecture Notes in Applied and Computational Mechanics 75

Simon R. Eugster

# Geometric Continuum Mechanics and Induced Beam Theories

Springer

Simon R. Eugster

# Geometric Continuum Mechanics and Induced Beam Theories

Springer

Simon R. Eugster
Institute for Nonlinear Mechanics
University of Stuttgart
Stuttgart
Germany

ISSN 1613-7736 ISSN 1860-0816 (electronic)
Lecture Notes in Applied and Computational Mechanics
ISBN 978-3-319-16494-6 ISBN 978-3-319-16495-3 (eBook)
DOI 10.1007/978-3-319-16495-3

Library of Congress Control Number: 2015933617

Springer Cham Heidelberg New York Dordrecht London


Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

# Lecture Notes in Applied and Computational Mechanics

Volume 75

**Series editors**

*About this Series*

This series aims to report new developments in applied and computational mechanics—quickly, informally and at a high level. This includes the fields of fluid, solid and structural mechanics, dynamics and control, and related disciplines. The applied methods can be of analytical, numerical and computational nature.

More information about this series at http://www.springer.com/series/4623

# Preface

In the last centuries continuum mechanics developed from a theory treating very specific problems to a general theory suitable for many applications. Continuum mechanics started with the description of one-dimensional continua where Euler's elastica is maybe its most famous problem. With the seminal work of Cauchy on the existence of the stress tensor in a three-dimensional continuum, the foundations of modern continuum mechanics have been laid down. After a century with the paradigm of infinitesimal deformations and linear elastic material laws, the second half of the twentieth century has been dominated by finite strain theories with large deformations and nonlinear material laws. Especially with the emergence of the computer and its fast rising power, to date, it is possible to treat more complex mechanical behavior than ever before. Nevertheless, an axiomatic consideration of continuum mechanics together with an appropriate mathematical framework is still a major challenge. The foundations of mechanics deal with the identification of the fundamental objects and the postulation of its principles. Due to the high level of abstraction, the mathematical discipline of intrinsic differential geometry seems to be best suited for the description of continuum mechanics. Step-by-step, additional mathematical structure can be introduced and motivated by the underlying physics. Without specifications of constitutive laws, geometric continuum mechanics is on the one hand coordinate independent and on the other hand a priori metric independent. Since a geometric continuum mechanics generalizes the well-established objects of the classical theories, every single object has to be rethought and evaluated if it is fundamental or not.

This book is intended to make the reverse direction of the historical development. It starts with an attempt of geometric continuum mechanics where body and physical space are assumed to be smooth manifolds. Combining the mechanical principles of Paul Germain from the 1970s with an intrinsic differential geometric description of continuum mechanics of Reuven Segev of the 1980s, the principle of virtual work emerges as the fundamental principle of continuum mechanics. In the second part of the book, the classical model of the physical space, the three-dimensional Euclidean space, is assumed and induced beam theories are treated as an application of continuum mechanics. Then it is possible to consider a beam as a continuous body with a constrained position field guaranteed by a perfect constraint stress field. Defining a constrained position field and applying the restricted kinematics to the principle of virtual work of a continuous body, the constraint stresses are eliminated due to the principle of d'Alembert–Lagrange and the weak variational form of an appropriate beam theory is induced directly. This induced approach to beam theory relates the point of view of beams as generalized one-dimensional continua to the theory of continuous bodies. In this work all classical beam theories, in which the cross sections remain rigid and plain, are presented. Additionally, augmented beam theories allowing for cross section deformations are derived using the very same procedure. All theories are suitable for large displacements and large rotations. The obtained weak variational forms of the appropriate beam theories serve then as the basis for the numerical implementation by finite elements.

The work presented in this book has been carried out during my time as research assistant at the Center of Mechanics at the ETH Zurich and appeared as doctoral thesis with the title "On the Foundations of Continuum Mechanics and its Application to Beam Theories". I was accompanied by many people whom I would like to thank for their kind support of my work. I am very thankful to my supervisor Prof. Dr.-Ing. Dr.-Ing. habil. Christoph Glocker for supporting and guiding my research. I have got the opportunity to delve into the very foundations of mechanics and by the way to improve my mathematical background enormously. His distinct idea of mechanics, based on the principle of virtual work as its fundamental principle, has always been a clear guideline to my work. Special thanks go to Prof. Dr. ir. habil. Remco Leine who has taught me the art of academic writing, has been an ideal of how to present research results, and has always been a critical voice in my research. I am looking forward to an intensive time as "Akademischer Rat" working together with him at the Institute for Nonlinear Mechanics at the University of Stuttgart. Many thanks go to Dr.-Ing. O. Papes who hooked me as a student to continuum mechanics and who infiltrated my mind by the concept of a geometric description of continuum mechanics. Finally, I would like to thank my family and friends for their support and continuous encouragement.

Stuttgart, December 2014 Simon R. Eugster

# Contents

## Part II Induced Beam Theories

# Chapter 1
# Introduction

This monograph is concerned with fundamental questions on the foundations of continuum mechanics and its application to beam theories. It does not pretend to be in any way 'complete', but merely serves as a discussion about novel approaches applied to these very classical fields of mechanics.

This first chapter starts with a short introduction and motivation for this book. Subsequently, Sect. 1.2 sheds some light on the virtual work in mechanics. After a literature survey in Sect. 1.3, the aim and scope of the work is presented in Sect. 1.4. An outline of the monograph is given in Sect. 1.5.

## 1.1 Motivation

One of the main goals of mechanics is the description and the prediction of the motion of mechanical devices, machines and mechanical processes. To meet this aim, abstract mechanical theories are formulated, thereby applying concepts from mathematical science. In such a determinism, a strict separation between reality and mathematical abstraction, called the model, has to be considered. The modeling process, being the procedure of the mathematical abstraction, is an interaction between the choice of the assumed mathematical structure and the description of observations in the real world within this mathematical framework. Hence, a mechanical theory can be developed on different levels of mathematical abstraction. The higher the level of mathematical abstraction, the less mathematical objects are involved and the more general a mechanical theory is. By increasing the level of abstraction in a mechanical theory, we try to extract the essential mechanical objects. An important step on that route of abstraction is the description of mechanics with as little mathematical structure as necessary, to recognize the fundamental laws of mechanics. There exists a vast amount of specific mechanical theories in which many assumptions on the kinematics of the system and on constitutive level are taken. For instance, we may distinguish between rigid body mechanics, beam theories, shell



S.R. Eugster, *Geometric Continuum Mechanics and Induced Beam Theories*,
Lecture Notes in Applied and Computational Mechanics 75,
DOI 10.1007/978-3-319-16495-3_1

theories, theory of elasticity, theory of fluids, finite degree of freedom mechanics to name a few. A fundamental question emerges: does a mechanical theory on a high level of abstraction exist which is able to induce these specific theories? The question immediately asks for the assumptions and concepts to arrive in a rigorous way at all these specific theories. The question of the embedding of well-known specific theories in a more general mechanical theory is one of the major challenges of modern classical mechanics. Such an embedding of theories leads to a more compact formulation of the vast field of classical mechanics. It leads to a deeper understanding of mechanics and will eventually allow treating more complex mechanical systems. This is what can be understood as scientific progress.

## 1.2 The Virtual Work

A rather novel insight in analytical mechanics is that the virtual work of a mechanical system is invariant with respect to the change of coordinates. This is directly related to the fact that there is a (coordinate free) differential geometric definition of the virtual work. To explain the basic idea, consider the case of finite degree of freedom mechanics, where the configuration manifold fully describes the kinematic state of the mechanical system. A generalized virtual displacement is a tangent vector of the configuration manifold. A covector of the configuration manifold as an element of the cotangent space constitutes a generalized force. The virtual work is defined as the real number obtained by the evaluation of a generalized force acting on a generalized virtual displacement. This geometric definition of the virtual work is completely free of any choice of coordinates and does not require any further geometric structure such as a metric. With the geometrical point of view in mind, the determination of the configuration manifold, being the kinematic description of the mechanical system, induces the space of generalized forces of the mechanical system. In a nutshell, the choice of kinematics defines, in the sense of duality, what kind of forces we may expect.

An illustrating example is a moving particle in the Euclidean three-space, where the very same space corresponds to the configuration manifold of the particle. Consequently, the generalized forces are elements of the cotangent space of the Euclidean three-space. In the Euclidean three-space there exist two important isomorphisms. One isomorphism is a canonical isomorphism between the tangent space and the Euclidean three-space. The second isomorphism is the isomorphism between tangent and cotangent space induced by the Euclidean metric. Using both isomorphisms, a generalized force on the particle can be identified with an element from the Euclidean three-space. This corresponds to the very classical understanding of a force as a geometric object from the Euclidean three-space satisfying the parallelogram law. As a side note, it is meaningless to speak of such thing as a couple of the particle, since there is no kinematic counterpart in the description of a particle.

Being the invariant object in mechanics, the virtual work almost naturally emerges as a central element in the postulation of the fundamental laws of mechanics.

The virtual work of a mechanical system is the sum of the virtual work contributions of all forces of the mechanical system. The principle of virtual work, stated as an axiom, claims that the virtual work of a mechanical system has to vanish for all virtual displacements. Hence, the principle of virtual work as a fundamental mechanical law is a coordinate free and metric independent formulation. Introducing more geometric structure as e.g. a metric, it is possible to formulate constitutive laws which relate force quantities with kinematic quantities and to arrive at more specific mechanical theories. For instance, a metric is required to define the strain of a continuous body which is necessary for the formulation of a material law. Another example is the formulation of the dynamics of a particle moving in the Euclidean three-space. The linear relation between the velocity of the particle and the linear momentum needs a metric of the space and the mass of the particle as a proportionality factor. Thus, from a differential geometric point of view, the linear momentum as "as mass times velocity" can be considered as an assumption on constitutive level.

In computational mechanics for infinite dimensional systems, the principle of virtual work in the form of weak variational forms is a fully accepted concept. It is used to perform existence and uniqueness proofs on the one hand, and to develop numerical schemes on the other hand. As a variational formulation, the principle of virtual work provides the only possibility within classical mechanics to mathematically define perfect bilateral constraints. The latter is done in form of a variational equality, known as the principle of d'Alembert–Lagrange, which puts the constraint forces into the annihilator space of the admissible virtual displacements. The concept of perfect constraints is omnipresent in each branch of mechanics and is quintessential to induce more specific theories from a general mechanical theory.

Many specific mechanical theories can be considered as special cases of the theory of continuous bodies. Rigid body mechanics, for instance, is the dynamics of a continuous body whose deformation is constrained such that the position field of the body can be described by a displacement of one material point of the body and a rotation of the body. Hence, the rigid body can be considered as a constrained continuous body. As discussed above, the principle of virtual work as a variational formulation is the only way to treat perfect bilateral constraints. Consequently, to induce a specific mechanical theory from the theory of a continuous body by imposing further constraints on the mechanical system, a variational formulation of the dynamics of a continuous body is inevitable.

In order to obtain an intrinsic theory of a continuous body in variational form, we have to use the concepts of analytical mechanics, where the forces are induced by the choice of the kinematics of the mechanical system. Before starting with a play, the actors and the scene have to be determined. Here, the body plays the role of a single actor and the scene is given by the model of the physical space. The play, i.e. how the body performs on the scene, corresponds to the admissible configurations of the body in the physical space. Using appropriate definitions of the body and the physical space, the set of all maps of the body into the physical space build an infinite dimensional manifold, called configuration manifold. This configuration manifold induces as in the finite dimensional setting the space of forces in the sense of duality. Applying the principle of virtual work together with further assumptions, which forces are involved, how these forces are represented and what their virtual work contribution is, this leads us directly to the fundamental law of a continuous body in a variational setting.

The section is closed with a list of several reasons why a general mechanical theory should be formulated variationally by the principle of virtual work.

- The space of forces of a mechanical system is induced by its kinematics. Hence, the forces cannot be defined regardless of the underlying kinematics.
- Set-valued force laws, including perfect bilateral constraints, can only be formulated variationally. This argument follows directly the slogan of P.D. Panagiotopoulos "In mechanics, there are forces and force laws".
- Many specific mechanical theories can be obtained by constraining the position field of a more general theory. To treat the perfect bilateral constraints, a variational formulation is inevitable.
- The most successful numerical methods, as e.g. finite element methods, rely on variational formulations.
- Under special assumptions on the mechanical system, variational problems and energy methods are directly obtained from the principle of virtual work.
- An intrinsic differential geometric formulation of mechanics requires the virtual work. Insofar, a more general definition of a body and the physical space is possible. Thus, the physical space is not restricted to be a Euclidean space and can be modeled, for instance, as a space-time vector bundle.

## 1.3 Literature Survey

In this section, a short literature survey on the foundations of continuum mechanics and on beam theories is given. To understand some developments in the foundations of continuum mechanics, it is tried to bring the literature on this field into a rough historical context. The survey on beam theories is merely intended to give some references which might be helpful in getting more detailed information.

### *1.3.1 Foundations of Continuum Mechanics*

Over the past two centuries continuum mechanics has become one of the cornerstones of classical mechanics. Evidently, there exist an immense and unmanageable number of publications on the foundations of continuum mechanics. After the celebrated theorem about the existence of a stress tensor of Cauchy [1] and the derivation of Cauchy's first law of motion [2], the 19th century was to a large extent occupied with continuum mechanics for very specific material laws. The theory of elasticity, i.e. continuum mechanics for solids with infinitesimal deformations and linear elastic material laws, was the predominated paradigm for solids. This very specific theory allows to find analytic solutions for many problems. Hence, the theory of linear elasticity has been the basis to further develop the theory of strength of materials.

For a detailed historical overview of continuum mechanics in the 20th century, we refer to Maugin [3]. In the first part of this century, there has been an increase in popularity in the description of the behavior of solids undergoing finite deformations as variational problems. This trend is manifested in a series of publications such as Murnaghan [4], Reissner [5] or Doyle and Ericksen [6]. The drawback of a formulation of continuum mechanics as a variational problem is, that the possible material laws of the continuum are restricted to the very specific subset of hyperelastic material laws. This drawback has been eliminated in the seminal treatise of Truesdell and Toupin [7] in which the theory of classical field theories is based on the balance of linear and angular momentum. The treatment therein has been mainly influenced by the system of axioms formulated in Noll [8, 9], a former student of Truesdell. The balance of linear and angular momentum, completed by the balance of energy and the conservation of mass, are generally referred to as the balance laws. Soon after the publication of the classical field theories, the theory of continuum mechanics has been enriched in Truesdell and Noll [10], completing the former work by an extensive treatise on material laws. The influence of Truesdell and Noll on continuum mechanics has led to a wealth of textbooks on continuum mechanics which follow the very same philosophy, e.g. Malvern [11], Gurtin [12], Chadwick [13], Holzapfel [14], Liu [15], Spencer [16], Dvorikin [17]. For a treatment in curvilinear coordinates, we refer to Green and Zerna [18], Ogden [19], Ciarlet [20] and Başar and Weichert [21]. The approach of Truesdell and Noll to continuum mechanics is revealed in the list of contents of the very technical treatise on rational continuum mechanics [22]: "I. Bodies, Forces, and Motions", "II. Kinematics", "III. The Stress Tensor" and "IV. Constitutive Relations". Very outstanding in this approach to continuum mechanics is the strong division between balance laws and constitutive laws. The attitude of Truesdell concerning variational principles is clarified in [7], Par. 231, where he distances himself from variational principles as fundamental equations of mechanics and regards them merely as derivative and subservient to the balance laws. Even stronger words can be found in [23] where he claims, that Lagrange has misunderstood or neglected general principles and concepts of mechanics.

According to [7], the first application of the virtual work to a continuum can be found in Piola [24]. Eighty years later, Hellinger [25] based continuum mechanics on the principle of virtual work and emphasized the benefits of the invariance of the virtual work. Therein, the virtual work of the continuum is formulated as the duality between the 1st Piola-Kirchhoff stress and the gradient of the virtual displacements. As one of the last classic books on theoretical mechanics in the German literature, also Hamel [26] applied the principle of virtual work to continuous bodies.

In the French literature there has been a renaissance of the concept of virtual work, or more precisely "Les puissances virtuelles", in the context of continuum mechanics induced by the publications of Germain [27–29]. Therein, first gradient and second gradient theories as well as continua with microstructure have been applied by the postulation of virtual work principles. One important contribution is the "Axiom of Power of Internal Forces" which corresponds to the variational formulation of the law of interaction. Another important contribution is the recognition, that also external stress contributions can be considered in a gradient theory. A rather mathematical approach to the idea of duality in mechanics has been developed in Nayroles [30]. A discussion about the stress tensor as dual object to a strain distribution has been given by Moreau [31]. The treatment of continuum mechanics using the principle of virtual work as its fundamental law of mechanics can be found in introductory textbook form by Germain [32] and Salençon [33].

The principle of virtual work is often used, when a coupling between different theories is demanded. Maugin [34] has formulated a coupling between the theory of electromagnetism and mechanics formulating a virtual work principle. A coupling between non-equilibrium thermodynamics and mechanics has been formulated by Biot [35]. In the publication of Del Piero [36], the internal virtual work is deduced from an invariance of the virtual work of the external force under change of observer. The equivalence of the principle of virtual work and the integral laws under certain regularity conditions has been shown in Antman and Osborn [37].

Noll [9] already recognized the importance of a differential geometric point of view on continuum mechanics and introduced the idea to regard a body as a smooth manifold. In the well-known treatise on the foundations of elasticity, Marsden and Hughes [38] have formulated body and space as Riemannian manifolds. The fundamental law of covariant elasticity has been considered as an invariance principle of energy, proposed in a non-differential geometric setting by Green and Rivlin [39]. An application of the covariant theory to solids, rods and plates has been treated by Simo et al. [40]. In Kanso [41] a new differential geometric interpretation of the stress tensor as a covector-valued differential two-form is given. A concise differential geometric consideration of the kinematics of the body and the space as manifolds has been presented by Aubram [42].

A formulation of continuum mechanics in an intrinsic differential geometric setting is discussed in the seminal work of Segev [43]. In the sense of analytical mechanics, forces of $n$th gradient theories are defined by duality. Using the concepts of jet-bundles and covariant derivatives, force representations for the gradient theories have been found. The virtual work principle is formulated as a mathematical compatibility condition between a force of the continuum and a stress representation. When the physical space is equipped with a connection, the variational stress of a first gradient theory is obtained as a linear map of the covariant derivative of the virtual displacement field to a volume form of the body. To date, perhaps the only existing work on intrinsic differential geometric formulation of continuum mechanics stems from Segev and his supervisor Epstein. The first steps in the development of this theory can be found in Epstein and Segev [44] and in the dissertation of Segev [45]. Segev [46] gives an application of the intrinsic theory to the special case, where the physical space is assumed as $\mathbb{R}^3$ and the body as a closed subset of the former. More explanations and focus on the fundamental questions of the intrinsic theory, can be found in [47–49]. An application of the theory to micro-structure has been presented in [50]. In a recent review article Segev [51] summarizes most of his publications. Due to the high level of abstraction in the intrinsic formulation, it is possible to contribute also in completely different fields, as the application to general relativity of [52] shows.

Due to the relaxation of the continuity assumptions and due to the generalization of the concept of stress, completely new fields, such as fractal mechanics [53], have been developed. Recently, an introductory textbook on the geometric understanding of continuum mechanics has been published by Epstein [54].

### *1.3.2 Beam Theory*

There exists a vast amount of treatises on the topic of beam theory. A very classical treatment of the mathematical theory of elasticity with application to beams is given by Love [55]. The beam equations are obtained by applying the balance of linear and angular momentum at an infinitesimal beam element. A textbook with plenty of applications and examples on beams is Sokolnikoff [56]. Villaggio [57] introduces beams on the one hand as an approximation of the three-dimensional elastic theory and on the other hand as directed curves. Classical linear beam theories are discussed in Bauchau and Craig [58]. For linear theories of beams, including beams with warping fields, we refer to Hjelmstad [59]. An extensive treatise on nonlinear beam theories is given in Antman [60], where almost any possible interpretation of beams is discussed. Outstanding is the chapter on generalized beam theories which relies on [61], in which beams are considered as constrained continuous bodies. A concise introduction to intrinsic special Cosserat beam theory is given by Ballard and Millard [62]. A discussion about two-director Cosserat beams also dealing with beam constitutive laws is part of Rubin [63]. A theory of beams deduced at an infinitesimal beam element and reformulated to a virtual work expression can be found in Wempner [64]. For a textbook including more involved cross section deformations we refer to Hodges [65].

The plane and linear Timoshenko beam has originally been developed in Timoshenko [66, 67]. The treatment of the same kinematical assumption for large displacements but small strains has been given for the plane and the spatial case by Reissner [68, 69]. Another derivation for the spatial Timoshenko beam has been obtained by Simo [70]. Considering the Timoshenko beam as a constrained continuum, Clerici [71] induces the weak and strong variational form of the Timoshenko beam from the virtual work principle of a three-dimensional continuous body. The same approach is proposed by Auricchio et al. [72].

The plane Euler–Bernoulli beam has been formulated as an induced theory by Epstein and Murray [73]. A spatial version is discussed in Hodges et al. [74]. The Kirchhoff beam, originating from Kirchhoff [75], is treated in a more modern version by Dill [76].

Augmented beam theories are theories, in which the cross sections are not restricted to remain plane and rigid. Classically, as proposed by Cosserat and Cosserat [77], such theories are formulated by intrinsic director theories, where the equations of motion are obtained by an invariance principle of a stated action. The theory of one-dimensional Cosserat media are included in Naghdi [78] and Cohen [79] and a theory of directed curves with further constraints is developed in