普通高等教育“十三五”规划教材

Linear Algebra

（线性代数英文版）

孙晓娟　编著

北京邮电大学出版社
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内 容 简 介

The main contents of *Linear Algebra* are matrices and determinants, systems of linear equations, eigenvalues and eigenvectors of square matrices, and quadratic form. Firstly, the concept of matrix is introduced, and then matrix arithmetic, determinants of square matrices, block matrices, invertible matrices, elementary matrices and rank of matrices are introduced. The second chapter introduces general solutions of linear systems, linear independence of vectors and the orthogonal basis of a vector space. The third chapter introduces eigenvalues and eigenvectors, and diagonalization of square matrices. Finally, the fourth chapter introduces the quadratic form and its matrix and how to change quadratic form into diagonal one.

Linear Algebra 的主要内容是矩阵和行列式、线性方程组、方阵的特征值和特征向量、二次型,共四个章节。第 1 章先引入矩阵的概念,而后介绍矩阵的基本运算和性质、矩阵的秩和逆、方阵的行列式运算及其性质;第 2 章介绍线性方程组的解、向量组的线性相关性、正交基;第 3 章介绍方阵的特征值与特征向量,以及方阵的相似对角化;最后,第 4 章介绍二次型及其矩阵和将二次型化为标准型的方法。

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Preface

"Linear algebra" is a branch of mathematics. The research objects of linear algebra are vectors, vector space (or linear space), linear equations. Vector space is an important subject in modern mathematics; therefore, linear algebra is widely used in abstract algebra and functional analysis. By analytic geometry, linear algebra can be expressed in detail. The theory of linear algebra has been generalized into the theory of operators. Because nonlinear models in scientific research can usually be approximated as linear models, linear algebra is widely applied to natural science and social sciences.

For students majoring in science and engineering, "linear algebra" is the basic course of mathematics. It is the essential basic course for other successor mathematics courses, such as probability theory and mathematical statistics, matrix theory, optimization and so on. Therefore, it is the most important part of the mathematics course. The teaching goal is to make students master the basic theoretical knowledge of linear algebra through systematic study and strict training, to cultivate strict logic thinking ability and reasoning ability, to have skilled operation ability and skill, to improve the mathematical model and to apply the linear generation to solve practical application problems. At the same time, we should cultivate students' scientific and rigorous thinking habits and meticulous work style in the learning process.

Different from the other Linear Algebra textbooks, this textbook is mainly for overseas students and Chinese students who have plans to go abroad. This textbook is relatively simple in content setting, but generally contains the main contents of linear algebra. Therefore, this textbook is very easy to understand and learn for overseas students and Chinese students who have plans to go abroad.

Limit to the level of the author, deficiencies and errors of this book are inevitable. I sincerely hope that readers will make comments and corrections.

Contents

Chapter 1

Matrices and Determinants

Matrix is an important mathematical tool. It is not only one of the main research content of linear algebra, but also the basis to learn the later chapters of this book. The matrix has been widely used in various fields of science and technology. This chapter introduces the concept of matrix, matrix arithmetic, matrix algebra and the determinant of the matrices. Then we will discuss the elementary transformation of matrices and elementary matrices, rank of matrices, etc.

1.1 Matrices

Example 1.1 In transportation, a product is transported from m origin places x_1, $x_2, \cdots, x_m$ to n product sales $y_1, y_2, \cdots, y_n$, the number of available transport can be represented by the following table:

Place of Origin	Place of Sale			
	y_1	y_2	$\cdots$	y_n
x_1	a_{11}	a_{12}	$\cdots$	a_{1n}
x_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
x_m	a_{m1}	a_{2n}	$\cdots$	a_{mn}

The number a_{ij} in the table indicates the quantity of the product transported from the origin place x_i to the sale place y_j, this table $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, which is arranged in a certain order, could present the transport scheme of the product.

Example 1.2 A system of linear equations

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1, \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2, \\ \qquad\qquad\vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m, \end{cases} \tag{1.1}$$

we can arrange the coefficients of this system into a table according to the original order

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

meanwhile, the right-hand side of the system could also be arranged into a table with the original order

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

With these two tables, the system of equations (1.1) is completely determined.

Such a rectangular array of numbers is often used in areas, such as natural sciences and engineering techniques. In mathematics, this type of table is called as matrix.

Definition 1.1 A rectangular array

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

of scalars a_{ij} $(i=1,2,\cdots,m;j=1,2,\cdots,n)$ is called a matrix of m rows and n columns. The pair of numbers m and n is called the size of the matrix. The scalar a_{ij}, called the i-th row and j-th column element of the matrix or ij-element, appears in row i and column j ($i=1,2,\cdots,m$, $j=1,2,\cdots n$). The cross position of the i-th row and j-th column is referred to as (i,j).

Matrices are usually symbolized using upper-case letters such as A,B,C. An $m\times n$ matrix $\mathbf{A}$ can be recorded as $\mathbf{A}=(a_{ij})_{m\times n}$ or $\mathbf{A}_{m\times n}$.

Remark 1 An $m\times n$ matrix is a table not a number which consists of $m\times n$ scalars, and only 1×1 matrix is a number.

Remark 2 The matrix whose elements are real numbers is called as real matrix. The matrix whose elements are complex numbers is called as complex matrix. In this book, real matrices are mainly concerned.

When $m=1$, we call

$$\boldsymbol{A}=(a_1 \quad a_2 \quad \cdots \quad a_n)$$

the row matrix or $1\times n$ matrix. $1\times n$ matrix is also called as n-dimension row vector. To avoid the confusion among elements, row matrix is also recorded as

$$\boldsymbol{A}=(a_1,a_2,\cdots,a_n).$$

When $n=1$, we call

$$\boldsymbol{B}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

the column matrix or $m\times 1$ matrix. $m\times 1$ matrix is called as m-dimension column vector.

Vectors are special matrices, which are of importance. Here, the dimension of vectors is the number of elements. Row matrix and column matrix are also symbolized using lower-case letters $\boldsymbol{a},\boldsymbol{b},\cdots,\boldsymbol{x},\boldsymbol{y},\cdots$.

When $m=n$, we call

$$\boldsymbol{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

the square matrix. Namely, a square matrix is a matrix with the same number of rows as columns. An $n\times n$ square matrix is said to be of order n and is sometimes called an n-th order square matrix. The elements $a_{11},a_{22},\cdots,a_{nn}$ on the diagonal of n-th order square matrix are called the diagonal elements of the square matrix; The diagonal or main diagonal of $\boldsymbol{A}$ is consists by these elements.

The matrix whose elements are zero is called a zero matrix, which is symbolized as $\boldsymbol{0}_{m\times n}$ or $\boldsymbol{0}$, that is

$$\boldsymbol{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Example 1.3

(a) The rectangular array $\boldsymbol{A}=\begin{pmatrix}1 & 3 & 0\\3 & 8 & 1\end{pmatrix}$ is a 2×3 matrix.

(b) The 2×4 zero matrix is the matrix $\mathbf{0}=\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$.

Special type of square matrices

1. Diagonal matrices

The matrix like

$$\boldsymbol{\Lambda}=\begin{pmatrix}a_{11} & 0 & \cdots & 0\\0 & a_{22} & \cdots & 0\\\vdots & \vdots & & \vdots\\0 & 0 & \cdots & a_{nn}\end{pmatrix}\text{ or }\boldsymbol{\Lambda}=\begin{pmatrix}a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn}\end{pmatrix}$$

is called diagonal matrix. Namely, a square matrix is diagonal if its nondiagonal elements are all zeros. However, some or all the a_{ii} maybe zero. It can also be represented by diag$(a_{11},a_{22},\cdots,a_{nn})$. For example,

$$\begin{pmatrix}3 & 0 & 0\\0 & -2 & 0\\0 & 0 & 1\end{pmatrix},\quad\begin{pmatrix}4 & 0\\0 & 3\end{pmatrix},\quad\begin{pmatrix}6 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 10 & 0\\0 & 0 & 0 & -8\end{pmatrix}$$

are diagonal matrices, which may be represented, respectively, by

$$\text{diag}(3,\ -2,\ 1),\ \text{diag}(4,\ 3),\ \text{diag}(6,\ 0,\ 10,\ -8)$$

2. Scalar matrices

When an n-th order square matrix with a's on the diagonal and 0's elsewhere, we call it the scalar matrix. The n-th order scalar matrix has the form as follows:

$$\begin{pmatrix}a & 0 & \cdots & 0\\0 & a & \cdots & 0\\\vdots & \vdots & & \vdots\\0 & 0 & \cdots & a\end{pmatrix}_{n\times n}\text{ or }\begin{pmatrix}a & & & \\ & a & & \\ & & \ddots & \\ & & & a\end{pmatrix}_{n\times n}.$$

Especially $a=1$, we call it the n-th order identity matrix. The n-th order identity matrix is symbolized as $\boldsymbol{E}_n$ or $\boldsymbol{I}_n$, that is

$$\boldsymbol{E}_n=\begin{pmatrix}1 & 0 & \cdots & 0\\0 & 1 & \cdots & 0\\\vdots & \vdots & & \vdots\\0 & 0 & \cdots & 1\end{pmatrix}\text{ or }\boldsymbol{E}_n=\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}.$$

In the absence of confusion, the identity matrix can be expressed in $\boldsymbol{E}$ or $\boldsymbol{I}$.

3. Upper and lower triangular matrices

The matrices like

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

are called upper triangular matrices and lower triangular matrices respectively.

1.2 Matrix Arithmetic

1.2.1 Equality

Definition 1.2 Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal, if they have the same size and the corresponding elements are equal, namely,

$$\boldsymbol{A}=\boldsymbol{B} \Leftrightarrow a_{ij}=b_{ij}, i=1,2,\cdots,m, j=1,2,\cdots,n.$$

Note that two matrices are not equal if they do not have the same size.

Example 1.4 Find a,b,c,d such that

$$\begin{pmatrix} a+2 & b-3 \\ 2c & 3a+d \end{pmatrix}=\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}.$$

By definition of equality of matrices, the four corresponding elements must be equal. Thus,

$$a+2=2, \quad b-3=1, \quad 2c=4, \quad 3a+d=5,$$

solving the above system of equations yields $a=0, b=4, c=2, d=5$.

1.2.2 Scalar Multiplication

Definition 1.3 The product of the matrix $\boldsymbol{A}$ by a scalar k, written $k \cdot \boldsymbol{A}$ or simply $k\boldsymbol{A}$, is the matrix obtained by multiplying each element of $\boldsymbol{A}$ by k. That is,

$$k\boldsymbol{A}=\begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}.$$

The scalar matrix $\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}_{n\times n}$ is the product of the scalar a and the identity matrix $\boldsymbol{E}_n$. For any matrix $\boldsymbol{A}$, $-\boldsymbol{A}=(-)\boldsymbol{A}=(-a_{ij})_{m\times n}$, which is called the negative matrix of $\boldsymbol{A}$.

Suppose $\boldsymbol{A},\boldsymbol{B}$ are $m\times n$ matrix, k,l are scalars, then we have

(1) Associative law: $(kl)\boldsymbol{A}=k(l\boldsymbol{A})=kl\boldsymbol{A}$;

(2) Distributive law: $k(\boldsymbol{A}+\boldsymbol{B})=k\boldsymbol{A}+k\boldsymbol{B}$, $(k+l)\boldsymbol{A}=k\boldsymbol{A}+l\boldsymbol{A}$.

1.2.3 Matrix Addition

Definition 1.4 Suppose that $\boldsymbol{A}=(a_{ij})_{m\times n}$, $\boldsymbol{B}=(b_{ij})_{m\times n}$ are two $m\times n$ matrices, the sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, written $\boldsymbol{A}+\boldsymbol{B}$, is the matrix obtained by adding corresponding elements from $\boldsymbol{A}$ and $\boldsymbol{B}$. That is,

$$\boldsymbol{A}+\boldsymbol{B}=(a_{ij}+b_{ij})_{m\times n}.$$

Note that if two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ do not have the same size, then they cannot do matrix addition.

Suppose $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ are $m\times n$ matrices, $\boldsymbol{0}=(0)_{m\times n}$ is zero matrix, then

(1) Commutative law: $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$;

(2) Associative law: $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$;

(3) $\boldsymbol{A}+\boldsymbol{0}=\boldsymbol{0}+\boldsymbol{A}=\boldsymbol{A}$;

(4) Cancellation law: $\boldsymbol{A}+\boldsymbol{C}=\boldsymbol{B}+\boldsymbol{C}\Leftrightarrow\boldsymbol{A}=\boldsymbol{B}$.

Obviously

$$\boldsymbol{A}+(-\boldsymbol{A})=(-\boldsymbol{A})+\boldsymbol{A}=\boldsymbol{0}.$$

And we can also define the matrix subtraction as

$$\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B})=(a_{ij}-b_{ij})_{m\times n}.$$

Namely, matrix subtraction $\boldsymbol{A}-\boldsymbol{B}$ is obtained by subtracting the corresponding elements from $\boldsymbol{A}$ to $\boldsymbol{B}$.

Example 1.5 If $\boldsymbol{A}=\begin{pmatrix} 1 & 0 & 3 \\ 1 & 5 & -4 \\ 6 & 2 & 0 \\ -7 & 9 & 3 \end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix} 5 & -2 & -4 \\ -5 & 6 & 3 \\ 0 & 3 & -1 \\ 8 & 7 & -7 \end{pmatrix}$, compute $4\boldsymbol{A}-\boldsymbol{B}$.

Solution:

$$4\boldsymbol{A}-\boldsymbol{B}=4\begin{pmatrix}1&0&3\\1&5&-4\\6&2&0\\-7&9&3\end{pmatrix}-\begin{pmatrix}5&-2&-4\\-5&6&3\\0&3&-1\\8&7&-7\end{pmatrix}$$

$$=\begin{pmatrix}4-5&0+2&12+4\\4+5&20-6&-16-3\\24-0&8-3&0+1\\-28-8&36-7&12+7\end{pmatrix}=\begin{pmatrix}-1&2&16\\9&14&-19\\24&5&1\\-36&29&19\end{pmatrix}.$$

Example 1.6 Given

$$\boldsymbol{A}=\begin{pmatrix}3&0&1\\9&7&-2\\5&-4&3\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}0&7&4\\5&1&3\\-2&-6&0\end{pmatrix}$$

and $\boldsymbol{A}-3\boldsymbol{X}=\boldsymbol{B}$, find $\boldsymbol{X}$.

Solution:

$$\boldsymbol{X}=\frac{1}{3}(\boldsymbol{A}-\boldsymbol{B})=\frac{1}{3}\begin{pmatrix}3&-7&-3\\4&6&-5\\7&2&3\end{pmatrix}=\begin{pmatrix}1&-\frac{7}{3}&-1\\\frac{4}{3}&2&-\frac{5}{3}\\\frac{7}{3}&\frac{2}{3}&1\end{pmatrix}.$$

1.2.4 Matrix Multiplication

Definition 1.5 Suppose $\boldsymbol{A}=(a_{ij})_{m\times s}$, $\boldsymbol{B}=(b_{ij})_{s\times n}$. Let $\boldsymbol{C}=(c_{ij})_{m\times n}$ is an $m\times n$ matrix which is composed of the $m\times n$ elements as follows

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{is}b_{sj}\quad(i=1,2,\cdots,m,j=1,2,\cdots,n).$$

Then the matrix $\boldsymbol{C}$ is the product of matrix $\boldsymbol{A}$ and $\boldsymbol{B}$, written as $\boldsymbol{C}=\boldsymbol{AB}$.

According to the definition above, $\boldsymbol{A}$ and $\boldsymbol{B}$ can multiply each other only if the number of columns of $\boldsymbol{A}$ equals the number of rows of $\boldsymbol{B}$. If $\boldsymbol{C}=\boldsymbol{AB}$, the number of rows of $\boldsymbol{C}$ equals the number of rows of $\boldsymbol{A}$, the number of columns of $\boldsymbol{C}$ equals the number of columns of $\boldsymbol{B}$. The i-th row and j-th column element of $\boldsymbol{C}$ equals the sum of the products of corresponding entries from the i-th row of $\boldsymbol{A}$ and the j-th column of $\boldsymbol{B}$, that is,

$$\begin{pmatrix}\cdots&\cdots&\cdots&\cdots\\a_{i1}&a_{i2}&\cdots&a_{js}\\\cdots&\cdots&\cdots&\cdots\end{pmatrix}\begin{pmatrix}\vdots&b_{1j}&\vdots\\\vdots&b_{2j}&\vdots\\\vdots&\vdots&\vdots\\\vdots&b_{2j}&\vdots\end{pmatrix}=\begin{pmatrix}&\vdots&\\\cdots&c_{ij}&\cdots\\&\vdots&\end{pmatrix},$$

where $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{is}b_{sj}$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

Example 1.7 Let $A=\begin{pmatrix}-1&3&2\\0&5&4\\1&1&2\end{pmatrix}$, $B=\begin{pmatrix}3&6\\0&-1\\5&2\end{pmatrix}$, compute AB.

Solution:

$$\begin{aligned}AB&=\begin{pmatrix}-1&3&2\\0&5&4\\1&1&2\end{pmatrix}\begin{pmatrix}3&6\\0&-1\\5&2\end{pmatrix}\\&=\begin{pmatrix}-1\times3+3\times0+2\times5&-1\times6+3\times(-1)+2\times2\\0\times3+5\times0+4\times5&0\times6+5\times(-1)+4\times2\\1\times3+1\times0+2\times5&1\times6+1\times(-1)+2\times2\end{pmatrix}\\&=\begin{pmatrix}7&-5\\20&3\\13&9\end{pmatrix}.\end{aligned}$$

Here A is a 3×3 matrix, and B is a 3×2 matrix. Because the number of column of B is not equal to the number of row of A, BA cannot be defined.

Therefore, in the matrix multiplication, we must pay attention to the matrix multiplication order. Note these facts:

(1) The matrix multiplication dissatisfies commutative law, namely $AB\neq BA$;

(2) If $AB=0$, we can not draw the conclusion that $A=0$ or $B=0$, which illustrates the matrix multiplication dissatisfies cancellation law.

However, if the matrix A and B satisfy $AB=BA$, A and B is commutative. At the same time, A and B are the same order square matrices.

Example 1.8 Given

$$A=\begin{pmatrix}1&0\\1&0\end{pmatrix},\quad B=\begin{pmatrix}0&0\\1&1\end{pmatrix}.$$

compute AB 和 BA.

Solution:

$$AB=\begin{pmatrix}1&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$BA=\begin{pmatrix}0&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\2&0\end{pmatrix}.$$

The matrix multiplication satisfies these laws:

(1) associative law: $(AB)C=A(BC)$;

(2) distributive law: $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}=\boldsymbol{AC}+\boldsymbol{BC}$, $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$;

(3) associative law of two kinds of multiplication: $k(\boldsymbol{AB})=(k\boldsymbol{A})\boldsymbol{B}=\boldsymbol{A}(k\boldsymbol{B})$, k is an arbitrary real number;

(4) $\boldsymbol{E}_m\boldsymbol{A}_{m\times n}=\boldsymbol{A}_{m\times n}$, $\boldsymbol{A}_{m\times n}\boldsymbol{E}_n=\boldsymbol{A}_{m\times n}$. (where $\boldsymbol{E}_m$, $\boldsymbol{E}_n$ are m-th order and n-th order identity matrix).

Example 1.9 Given $\boldsymbol{A}=\begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}$, find all matrices which are commutative with $\boldsymbol{A}$.

Solution: The matrices commutative with $\boldsymbol{A}$ must be the n-th order matrices, so we can suppose

$$\boldsymbol{X}=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}$$

is the matrix which is commutative with $\boldsymbol{A}$, then

$$\boldsymbol{AX}=\begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}=\begin{pmatrix}x_{11}+3x_{21} & x_{12}+3x_{22}\\ -x_{11}+2x_{21} & -x_{12}+2x_{22}\end{pmatrix},$$

$$\boldsymbol{XA}=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\begin{pmatrix}1 & 3\\ -1 & 2\end{pmatrix}=\begin{pmatrix}x_{11}-x_{12} & 3x_{11}+2x_{12}\\ x_{21}-x_{22} & 3x_{21}+2x_{22}\end{pmatrix}.$$

According to $\boldsymbol{AX}=\boldsymbol{XA}$, $x_{12}=-3x_{21}$, $x_{21}+x_{22}=x_{11}$ can be introduced, and x_{21}, x_{22} can be any value, then

$$\boldsymbol{X}=\begin{pmatrix}x_{21}+x_{22} & -3x_{21}\\ x_{21} & x_{22}\end{pmatrix}.$$

Suppose $\boldsymbol{A}$ is an n-th order square matrix, then we can define the power of square matrices as

$$\boldsymbol{A}^0=\boldsymbol{E},\boldsymbol{A}^1=\boldsymbol{A},\boldsymbol{A}^2=\boldsymbol{AA},\cdots,\boldsymbol{A}^k=\underbrace{\boldsymbol{AA}\cdots\boldsymbol{A}}_{k},$$

where k is a positive integer.

Suppose k, l is the arbitrary positive integer, then

$$\boldsymbol{A}^k\boldsymbol{A}^l=\boldsymbol{A}^{k+l},\quad (\boldsymbol{A}^k)^l=\boldsymbol{A}^{kl}.$$

The matrix multiplication dissatisfies commutative law, so that in general

$$(\boldsymbol{AB})^k\neq\boldsymbol{A}^k\boldsymbol{B}^k,k>1.$$

Furthermore, if $\boldsymbol{A}^k=\boldsymbol{0}(k>1)$, $\boldsymbol{A}=\boldsymbol{0}$ does not exist definitely.

Polynomials in the square matrix $\boldsymbol{A}$ are also defined. Specially, for any polynomial

$$f(x)=a_0x^m+a_1x^{m-1}+\cdots+a_{m-1}x+a_m,$$

$f(\boldsymbol{A})$ is defined to be the following matrix:

$$f(\boldsymbol{A})=a_0\boldsymbol{A}^m+a_1\boldsymbol{A}^{m-1}+\cdots+a_{m-1}\boldsymbol{A}+a_m\boldsymbol{E}_n.$$

Note that $f(\boldsymbol{A})$ is obtained from $f(x)$ by substituting the matrix $\boldsymbol{A}$ for the variable x and substituting the scalar matrix $a_m\boldsymbol{E}_n$ for the scalar a_m.

Example 1.10 Suppose $f(x)=x^2-4x+3$

$$\boldsymbol{A}=\begin{pmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{pmatrix},$$

compute $f(\boldsymbol{A})$.

Solution: $f(\boldsymbol{A})=\boldsymbol{A}^2-4\boldsymbol{A}+3\boldsymbol{E}$

$$=\begin{pmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{pmatrix}^2-4\begin{pmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{pmatrix}+3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}-1&3&-1\\-1&-1&3\\3&-1&-1\end{pmatrix}-\begin{pmatrix}4&4&-4\\-4&4&4\\4&-4&4\end{pmatrix}+\begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix}$$

$$=\begin{pmatrix}-2&7&-5\\3&6&7\\7&-5&6\end{pmatrix}.$$

1.2.5 Transpose of a Matrix

Definition 1.6 The transpose of a matrix $\boldsymbol{A}$, written $\boldsymbol{A}^{\mathrm{T}}$, is the matrix obtained by writing the columns of $\boldsymbol{A}$, in order, as rows. That is

$$\boldsymbol{A}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix},\ \text{then}\quad \boldsymbol{A}^{\mathrm{T}}=\begin{pmatrix}a_{11}&a_{21}&\cdots&a_{m1}\\a_{12}&a_{22}&\cdots&a_{m2}\\\vdots&\vdots&&\vdots\\a_{1n}&a_{2n}&\cdots&a_{mn}\end{pmatrix}.$$

For example, $\boldsymbol{A}=\begin{pmatrix}1&2&5\\6&4&3\end{pmatrix}$, then $\boldsymbol{A}^{\mathrm{T}}=\begin{pmatrix}1&6\\2&4\\5&3\end{pmatrix}$.

The transpose of a matrix satisfies the following laws:

(1) $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}=\boldsymbol{A}$;

(2) $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}}=\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{B}^{\mathrm{T}}$;

(3) $(k\boldsymbol{A})^{\mathrm{T}}=k\boldsymbol{A}^{\mathrm{T}}$;

(4) $(\boldsymbol{A}\boldsymbol{B})^{\mathrm{T}}=\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$, $(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k)^{\mathrm{T}}=\boldsymbol{A}_k^{\mathrm{T}}\boldsymbol{A}_{k-1}^{\mathrm{T}}\cdots\boldsymbol{A}_1^{\mathrm{T}}$.

We emphasize that, by (4), the transpose of a product is the product of the