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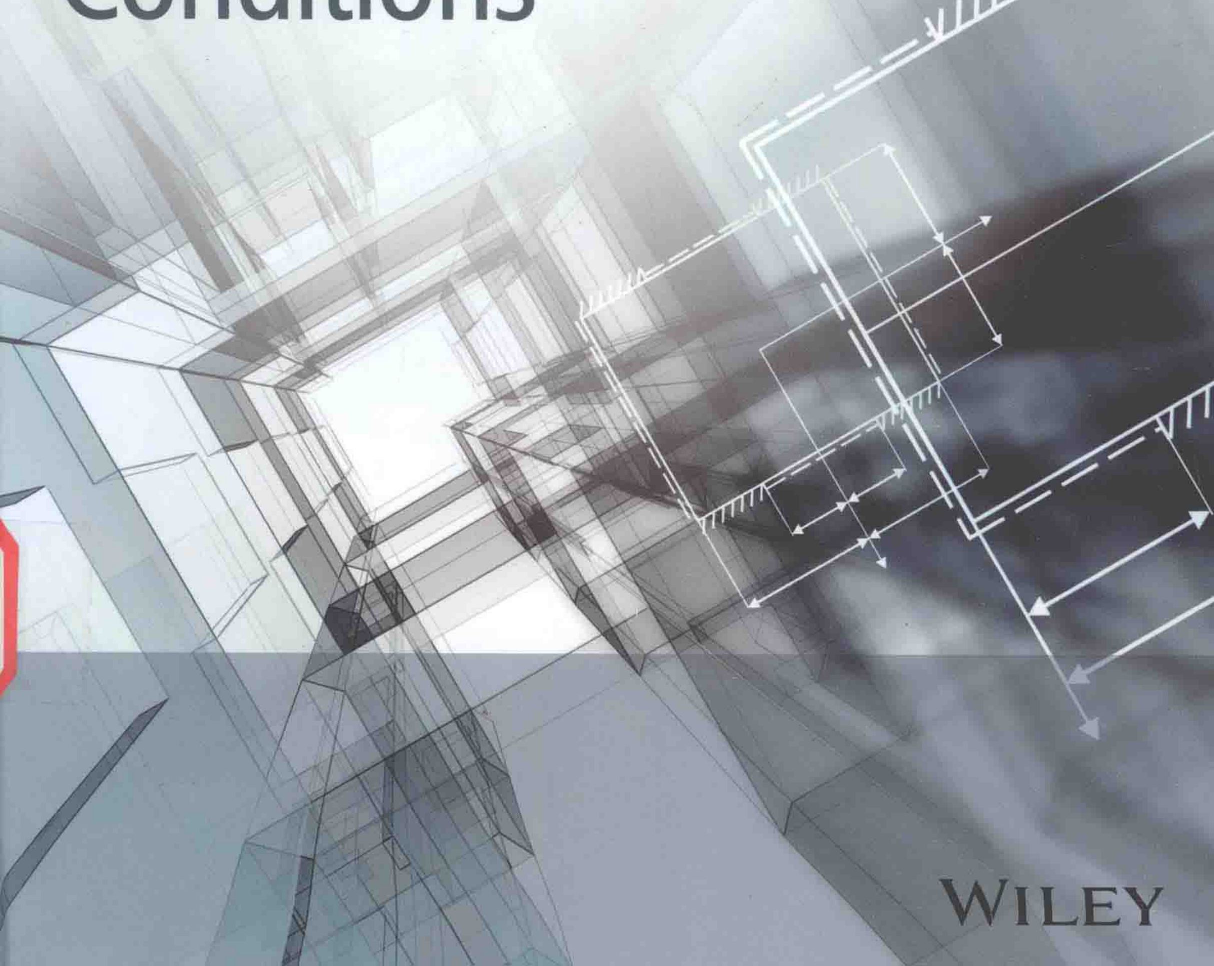

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ASYMPTOTIC METHODS IN THE THEORY OF PLATES WITH MIXED BOUNDARY CONDITIONS

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Preface

It is evident that plate structural elements are widely used in various branches of engineering. In industrial and civil engineering they serve as covers, working elements and parts of the various foundations; in the machine building they are elements of technological design. The above-mentioned construction members are intended to accommodate various static and dynamic excitations, and their strength, resistance and technical stability require increasing engineering expectations. In real constructions the boundary conditions are usually of a complicated character: free edge, clamping, elastic clamping, as well as various types of mixed boundary conditions. Similar conditions may occur in constructing various supports of different and mixed types. On the other hand, mixed boundary conditions may appear during the linkage of design structural members with the a use of various laps as well as intermittent welding. Furthermore, mixed boundary conditions may appear in supporting a plate beam on a nonsmooth surface. Finally, computation of plates with slits and cracks in many cases may be reduced to the computation of constructions with mixed boundary conditions. It should be emphasized that the computational scheme of a construction can be changed in the exploitation time due to the action of external loads (occurrence of corrosion and cracks, damage of part of a resistance support, etc.). In this case one may also expect a mixed boundary support, which was not predicted by the previous engineering analysis and design.

Nowadays, a wide spectrum of applications devoted to computations of the above-mentioned engineering objects can be solved by FEM (Finite Element Method). In practice, any problem can be solved via application of the appropriately chosen finite elements. However, it should be emphasized that FEM also suffers from a few drawbacks: it is rather difficult to estimate the validity of the FEM obtained results; in many cases instability in the vicinity of points occurs, where boundary conditions undergo changes, etc. This is why from the point of view of theory of plates and shells, as well as engineering practice, analytical approximate methods still play an important role in the study of a wide class of constructions with mixed boundary conditions. It seems that among analytical approaches, the asymptotic ones are most appropriate and successful in solving the problems discussed above.

It has recently been observed that asymptotic approaches again attract a big attention of many scientists in spite of the big development of numerical techniques [1]. The reason is mainly motivated by the intuition development of a researcher/engineer through asymptotic analysis. Even in a case where we are interested only in numerical solutions, a priori asymptotical analysis allows us to choose the most suitable numerical method and sheds light on usually disordered and largely numerically obtained material.

Moreover asymptotic analysis is extremely useful in providing the external value of parameters, where direct numerical computation meets serious difficulties in obtaining reliable results. This aspect of asymptotic methods has been well illustrated by the English scientist D.G. Crighton [2]: "*Design of computational or experimental schemes without the guidance of asymptotic information is wasteful at best, dangerous at worst, because of possible failure to identify crucial (stiff) features of the process and their localization in coordinate and parameter space. Moreover, all experience suggests that asymptotic solutions are useful numerically far beyond their nominal range of validity, and can often be used directly, at least at a preliminary product designs stage, for example, saving the need for accurate computation until the final design stage where many variables have been restricted to narrow ranges.*"

Since asymptotic methods play a key role in our book, the first part (Chapter 1) has been devoted to their description. We mainly rely on examples and avoid unnecessary generalizations. We have aimed to keep the book self-organized and discrete. In other words, the material in this book should be sufficient for the reader without need for supplementary material. In particular, we have focused on asymptotic approaches, which are either not well known or not well reported, such as the method of summation and construction of asymptotically equivalent functions, methods of small and large delta, homotopy perturbations method, etc.

Let us look briefly at the latter mentioned approach, which has recently been very popular. Its main idea is as follows. We introduce the parameter ε into either differential equations or boundary conditions in such a way that for $\varepsilon = 0$ we obtain the boundary value problem allowing us to find a simple solution, whereas for $\varepsilon = 1$ it gives the input boundary value problem. In the next step we apply the splitting method regarding ε, and in the finally obtained solution we put $\varepsilon = 1$. In other words, we apply a certain homotopic transformation. It is clear that this approach is not new, since it has already been successively applied by H. Poincaré [3] and A.M. Liapunov [4]. However, it has rarely been applied for many years because the obtained series are divergent in the majority of cases. This is why the homotopy perturbation method is supplemented by the effective summation method of the yielded series.

In particular, in order to solve this problem the application of the Padé approximation has been proposed in reference [5], which has been further developed in [6], [7], [8]. The method of boundary conditions perturbation also stands in the forefront of novel asymptotic development trends.

The second part of this book is devoted to application of the latter method to solve various problems of the theory of plates with mixed boundary conditions. Both free and forced vibrations of plates are studied, as well as their stress states and stability problems. One of the important benefits is that the results obtained are presented in simple analytical forms, and they can be directly used in engineering practice.

Furthermore, as we show, our analytical results possess high accuracy, since they have been compared either with known analytical or with numerical solutions.

Many of the results included this book have been obtained with the help of our colleagues, R.G. Barantsev, W.T. van Horssen, L.V. Kurpa, L.I. Manevitch, Yu.V. Mikhlin, V.O. Olevs'kyy, A.V. Pichugin, V.N. Pilipchuk, G.A. Starushenko, S. Tokarzewski, H. Topol, A. Vakakis, D. Weichert and we warmly acknowledge their input through numerous discussions and ideas exchanged at many conferences, meetings, congresses, symposia, etc.

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List of Abbreviations

ADM	Adomian decomposition method
AEF	asymptotically equivalent function
BC(s)	boundary condition(s)
BVP(s)	boundary value problem(s)
DE(s)	differential equation(s)
FEM	finite element method
HPM	homotopy perturbation method
l.h.s.	left hand side
LAE(s)	linear algebraic equation(s)
ODE(s)	ordinary differential equation(s)
PA	Padé approximants
PDE(s)	partial differential equation(s)
PS	perturbation series
r.h.s.	right hand side
SSS	stress-strain state
TPPA	two-point Padé approximants

Contents

1

Asymptotic Approaches

Asymptotic analysis is a constantly growing branch of mathematics which influences the development of various pure and applied sciences. The famous mathematicians Friedrichs [109] and Segel [217] said that an asymptotic description is not only a suitable instrument for the mathematical analysis of nature but that it also has an additional deeper intrinsic meaning, and that the asymptotic approach is more than just a mathematical technique; it plays a rather fundamental role in science. And here it appears that the many existing asymptotic methods comprise a set of approaches that in some way belong rather to art than to science. Kruskal [151] even introduced the special term "asymptotology" and defined it as the art of handling problems of mathematics in extreme or limiting cases. Here it should be noted that he called for a formalization of the accumulated experience to convert the art of asymptotology into a science of asymptotology.

Asymptotic methods for solving mechanical and physical problems have been developed by many authors. We can mentioned excellent monographs by Eckhaus [96], [97], Hinch [133], Holms [134], Kevorkian and Cole [147], Lin and Segel [162], Miller [188], Nayfeh [62], [63], Olver [197], O'Malley [198], Van Dyke [244], [246], Verhulst [248], Wasov [90] and many others [15], [20], [34], [71], [72], [110], [119], [161], [169], [173]-[175], [216], [222], [223], [250], [251]. The main feature of the present book can be formulated as follows: it deals with new trends and applications of asymptotic approaches in the fields of Nonlinear Mechanics and Mechanics of Solids. It illuminates developments in the field of asymptotic mathematics from different viewpoints, reflecting the field's multidisciplinaiy nature. The choice of topics reflects the authors' own research experience and participation in applications. The authors have paid special attention to examples and discussions of results, and have tried to avoid burying the central ideas in formalism, notations, and technical details.

1.1 Asymptotic Series and Approximations

1.1.1 Asymptotic Series

As has been mentioned by Dingle [92], theory of asymptotic series has just recently made remarkable progress. It was achieved through the seminal observation that application of

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asymptotic series is tightly linked with the choice of a summation procedure. A second natural question regarding the method of series summation emerges. It is widely known that only in rare cases does a simple summation of the series terms lead to satisfactory and reliable results. Even in the case of convergent series, many problems occur, which increase essentially in the case of a study of divergent series [64]. In order to clarify the problems mentioned so far, let us consider the general form of an asymptotic series widely used in physics and mechanics [65]:

$$\sum_{n=1}^{\infty} M_n \left(\frac{\varepsilon}{\varepsilon_0}\right)^n \Gamma(n+a), \tag{1.1}$$

where a denotes an integer, and Γ is a Gamma function (see [2], Chapter 6).

The quantity ε_0 is often referred to as a singulant, and M_n denotes a modifying factor. The sequence M_n tends to a constant for $n \to \infty$ and yields information on the slowly changed series part, whereas the constant ε_0 is associated with the first singular point of the initially studied either integral or differential equation linked to the series (1.1).

In what follows we recall the classical definition: a power type series is the asymptotic series regarding the function $f(\varepsilon)$, if for a fixed N and essentially small $\varepsilon > 0$, the following relation holds

$$\left| f(x) - \sum_{j=0}^{N} a_j \varepsilon^j \right| \sim O(\varepsilon^{N+1}),$$

where the symbol $O(\varepsilon^{N+1})$ denotes the accuracy order of ε^{N+1} (see Section 1.2).

In other words we study the interval for $\varepsilon \to 0$, $N = N_0$.

Although series (1.1) is divergent for $\varepsilon \neq 0$, its first terms vanish exponentially fast for $\varepsilon \ll \varepsilon_0$. This underscores an important property of asymptotic series, related to a game between decaying terms and factorial increase of coefficients. An optimal accuracy is achieved if one takes a smallest term of the series, and then the corresponding error achieves $\exp(-\alpha/\varepsilon)$, where $\alpha > 0$ is the constant, and ε is the small/perturbation parameter. Therefore, a truncation of the series up to its smallest term yields the exponentially small error with respect to the initial value problem. On the other hand, sometimes it is important to include the above-mentioned exponentially small terms from a computational point of view, since it leads to improvement of the real accuracy of an asymptotic solution [52], [53], [64], [65], [226], [230].

Let us consider the following Stieltjes function (see [65]):

$$S(\varepsilon) = \int_0^{\infty} \frac{\exp(-t)}{1+\varepsilon t} dt. \tag{1.2}$$

Postulating the approximation

$$\frac{1}{1+\varepsilon t} = \sum_{j=0}^{N} (-\varepsilon t)^j + \frac{(-\varepsilon t)^{N+1}}{1+\varepsilon t}, \tag{1.3}$$

and putting series (1.3) into integral (1.2) we get

$$S(\varepsilon) = \sum_{j=0}^{N} (-\varepsilon^j) \int_0^{\infty} t^j \exp(-t) dt + E_N(\varepsilon), \tag{1.4}$$

where

$$E_N(\varepsilon) = \int_0^\infty \frac{\exp(-t)(-\varepsilon t)^{N+1}}{1+\varepsilon t} dt. \quad (1.5)$$

Computation of integrals in Equation (1.4) using integration by parts yields

$$S(\varepsilon) = \sum_{j=0}^{N} (-1)^j j! \varepsilon^j + E_N(\varepsilon).$$

If N tends to infinity, then we get a divergent series. It is clear, since the under integral functions have a simple pole in the point $t = -1/\varepsilon$, therefore series (1.3) is valid only for $|t| < 1/\varepsilon$. The obtained results cannot be applied in the whole interval $0 \le t < \infty$.

Let us estimate an order of divergence by splitting the function $S(\varepsilon)$ into two parts, i.e.

$$S(\varepsilon) = S_1(\varepsilon) + S_2(\varepsilon) = \int_0^{1/\varepsilon} \frac{\exp(-t)}{1+\varepsilon t} dt + \int_{1/\varepsilon}^\infty \frac{\exp(-t)}{1+\varepsilon t} dt.$$

Since $1/(1+\varepsilon t) \le 1/2$ for $t > 1/\varepsilon$, the following estimation is obtained: $S_2(\varepsilon) < 0.5\exp(-1/\varepsilon)$.

Therefore, the exponential decay of the error is observed for decreasing ε, which is a typical property of an asymptotic series.

Let us now estimate an optimal number of series terms. This corresponds to the situation in which the term $t^{N+1}\exp(-t)$ in Equation (1.4) is a minimal one, which holds for $t = 1/(N+1)$. For $t \ge 1/\varepsilon$ we observe the divergence, and this yields the following estimation: $N = [1/\varepsilon]$, where [...] denotes an integer part of the number. The optimally truncated series is called the super-asymptotic one [65], whereas the hyperasymptotic series [52], [53] refers to the series with the accuracy barrier overcome. It means that after the truncation procedure one needs novel ideas to increase accuracy of the obtained results. Problems regarding a summation of divergent series are discussed in Chapters 1.3–1.5.

One may, for instance, transform the series part

$$S(\varepsilon) \approx \sum_{j=0}^{2N} (-1)^j j! \varepsilon^j \quad (1.6)$$

into the PA, i.e. into a rational function of the form

$$S(\varepsilon) \approx \frac{1 + \sum_{j=1}^{N} \alpha_j \varepsilon^j}{1 + \sum_{i=1}^{N} \beta_i \varepsilon^i}, \quad (1.7)$$

where constants α_j, β_i are chosen in a such a way that first $2N+1$ terms of the MacLaurin series (1.7) coincide with the coefficients of series (1.6). It has been proved that a sequence of PA (1.7) is convergent into a Stieltjes integral, and the error related to estimation of $S(\varepsilon)$ decreases proportionally to $\exp(-4\sqrt{N/\varepsilon})$.

The definition of an asymptotic series indicates a way of numerical validation of an asymptotic series [62]. Let us for instance assume that the solution $U_a(\varepsilon)$ is the asymptotic of the exact solution $U_T(\varepsilon)$, i.e.

$$E = U_T(\varepsilon) - U_a(\varepsilon) = K\varepsilon^\alpha.$$

One may take as U_T a numerical solution. In order to define α, usually graphs of the dependence $\ln E$ versus $\ln \varepsilon$ for different values of ε are constructed. The associated relations should be closed to linear ones, whereas the constant α can be defined using the method of least squares. However, for large ε the asymptotic property of the solution is not clearly exhibited, whereas for small ε values it is difficult to get a reliable numerical solution. Let us study an example of the following integral

$$I(\varepsilon) = \varepsilon e^{\varepsilon} \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt$$

for large values of ε. Although the infinite series

$$I(\varepsilon) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\varepsilon^n}$$

is divergent for all values of ε, series parts

$$I_M(\varepsilon) = \sum_{n=0}^{M} \frac{(-1)^n n!}{\varepsilon^n} \tag{1.8}$$

are asymptotically equivalent up to the order of $O(\varepsilon^{-M})$ with the error of $O(\varepsilon^{-M-1})$ for $x \to \infty$. In Figure 1.1 the dependence $\log E_M(\varepsilon)$ vs. $\log \varepsilon$, where $E_M(\varepsilon) = I(\varepsilon) - I_M(\varepsilon)$, is reported (curves going down correspond to decreasing values of $M = 1, \ldots, 5$).

It is clear that curve slopes are different. However, results reported in Table 1.1 of the least square method fully prove the high accuracy of the method applied.

Let us briefly recall the method devoted to finding asymptotic series, where the function values are known in a few points. Let a numerical solution be known for some values of the parameter ε: $f(\varepsilon_1), f(\varepsilon_2), f(\varepsilon_3)$. If we know a priori that the solution is of an asymptotic-type,

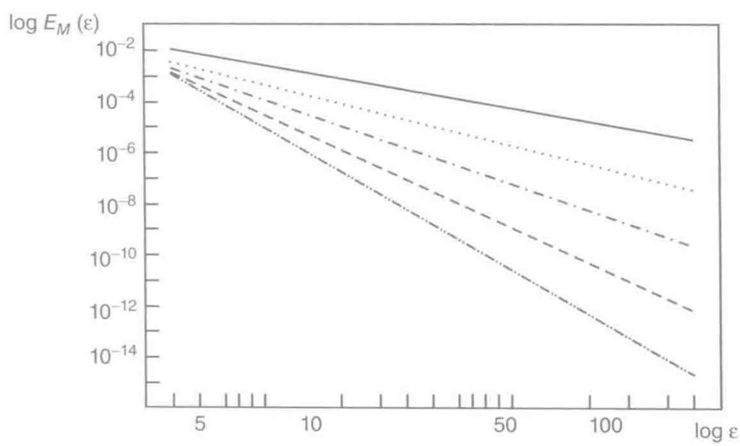

Figure 1.1 Asymptotic properties of partial sums of (1.8)

Table 1.1 Slope coefficient $\log E_M(\varepsilon)$ as the function of $\log \varepsilon$ defined via the least square method

$E_M(\varepsilon)$	$\varepsilon \in [5, 50]$	$\varepsilon \in [50, 200]$	$\varepsilon \in [200, 500]$	slope
1	−1.861	−1.972	−1.991	−2.0
2	−2.823	−2.963	−2.988	−3.0
3	−3.789	−3.954	−3.985	−4.0
4	−4.758	−4.945	−4.981	−5.0
5	−5.729	−5.937	−5.999	−6.0

and its general properties are known (for instance it is known that the series corresponds only to integer values of ε), then the following approximation holds

$$f(\varepsilon_i) = \sum_{i=0}^{3} \varepsilon^i a_i,$$

and the coefficients a_i can be easily identified. The latter approach can be applied in the following briefly addressed case. In many cases it is difficult to obtain a solution regarding small values of ε, whereas it is easy to find it for ε of order 1. Furthermore, assume that we know a priori the solution asymptotic for $\varepsilon \to 0$, but it is difficult or unnecessary to define it analytically. In this case the earlier presented method can be applied directly.

1.1.2 Asymptotic Symbols and Nomenclatures

In this section we introduce basic symbols and a nomenclature of the asymptotic analysis considering the function $f(x)$ for $x \to x_0$. In the asymptotic approach we focus on monitoring the function $f(x)$ behavior for $x = x_0$. Namely, we are interested in finding another arbitrary function $\varphi(x)$ being simpler than the original (exact) one, which follows $f(x)$ for $x \to x_0$ with increasing accuracy. In order to compare both functions, a notion of the order of a variable quantity is introduced accompanied by the corresponding relations and symbols.

We say that the function $f(x)$ is of order $\varphi(x)$ for $x \to x_0$, or equivalently

$$f(x) = O(\varphi(0)) \qquad \text{for} \qquad x \to x_0,$$

if there is a number A, such that in a certain neighborhood Δ of the point x_0 we have $|f(x)| \le A|\varphi(x)|$.

Besides, we say that $f(x)$ is the quantity of an order less than $\varphi(x)$ for $x \to x_0$, or equivalently

$$f(x) = o(\varphi(0)) \qquad \text{for} \qquad x \to x_0,$$

if for an arbitrary $\varepsilon > 0$ we find a certain neighborhood Δ of the point x_0, where $|f(x)| \le \varepsilon|\varphi(x)|$.

In the first case the ratio $|f(x)|/|\varphi(x)|$ is bounded in Δ, whereas in the second case it tends to zero for $x \to x_0$. For example, $\sin x = O(1)$ for $x \to \infty$; $\ln x = o(x^\alpha)$ for an arbitrary $\alpha > 0$ for $x \to \infty$.

Symbols $O(\dots)$ and $o(\dots)$ are often called Landau's symbols (see [62], [63]). It should be emphasized that Edmund Landau introduced these symbols in 1909, whereas Paul Gustav Heinrich Bachman had already done so in 1894. Sometimes it worthwhile to apply additional symbols introducing other ordering relations. Namely, if $f(x) = O(\varphi(x))$, but $f(x) \neq o(\varphi(x))$ for $x \to x_0$, then the following notation holds $f(x) = \tilde{O}(\varphi(x))$ for $x \to x_0$, where the symbol $\tilde{O}(\varphi(x))$ is called the symbol of the exact order (note that in some cases also the following symbol is applied $Oe(\varphi(x))$. If $f(x) = O(\varphi(x))$, $\varphi(x) = O(f(x))$ for $x \to x_0$, (it means that $f(x)$ asymptotically equals to $\varphi(x)$ for $x \to x_0$), which is abbreviated by the notation $f(x) \asymp \varphi(x)$ for $x \to x_0$. Recall that in some cases the symbol $\asymp$ is used. Asymptotic relations give rights for the existence of the numbers $a > 0$ and $A > 0$, where in the vicinity of the point x_0 the following approximation holds: $a|\varphi(x)| \leq |f(x)| \leq A|\varphi(x)|$.

Symbols $\tilde{O}$ and $\asymp$ might be expressed by O, o and are used only for a brief notation. One may distinguish the following steps while constructing an asymptotic approximation. In the beginning high (low) order estimations are constructed of the type $f(x) = O(\varphi(x))$. Usually this first approximation is overestimated, i.e. we have $f(x) = O(\varphi(x))$.

In order to improve this first approximation the following exact order is applied $f(x) = \tilde{O}(\varphi_0(x))$, and the following asymptotic approximation is achieved $f(x) \sim a_0\varphi_0(x)$. Carrying out this kind of a cycle, we may get the asymptotic chain $f(x) - a_0\varphi_0(x) \sim a_1\varphi_1(x)$, and go further with the introduced analysis. We say that the sequence $\{\varphi_n(x)\}$, $n = 0, 1, \dots$ for $x \to x_0$ is an asymptotic one, if $\varphi_{n+1}(x) = o(\varphi_n(x))$. For instance, the following sequence $\{x^n\}$ is an asymptotic one for $x \to 0$.

A series $\sum_{n=0}^{\infty} a_n\varphi_n(x)$ with constant coefficients is called an asymptotic one, if $\{\varphi_n(x)\}$ is an asymptotic sequence. We say that $f(x)$ has an asymptotic series with respect to the sequence $\{\varphi_n(x)\}$, or equivalently

$$f(x) \sim \sum_{n=0}^{N} a_n\varphi_n(x), \qquad N = 0, 1, 2, \dots, \tag{1.9}$$

if

$$f(x) = \sum_{n=0}^{m} a_n\varphi_n(x) + o(\varphi_m(x)), \qquad m = 0, 1, 2, \dots, N. \tag{1.10}$$

Let us investigate the uniqueness of the asymptotic series. Let the function $f(x)$ for $x \to x_0$ be developed into a series with respect to the asymptotic sequence $\{\varphi_n(x)\}, f(x) \sim \sum_{n=0}^{\infty} a_n\varphi_n(x)$. Then the coefficients a_n are defined uniquely via the following formula

$$a_n = \lim_{x\to x_0}\left[f(x) - \sum_{k=0}^{n-1} a_k\varphi_k(x)\right]\varphi_n^{-1}(x).$$

Observe that the same function $f(x)$ can be developed with respect to another sequence $\chi_n(x)$, for instance

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \quad \text{for} \quad x \to 0; \quad \frac{1}{1-x} \sim \sum_{n=0}^{\infty}(1+x)x^{2n} \quad \text{for} \quad x \to 0.$$

On the other hand, one asymptotic series may correspond to a few functions, for instance

$$\frac{1+e^{-1/x}}{1-x} \sim \sum_{n=0}^{\infty} x^n \qquad \text{for} \quad x \to 0.$$

In other words an asymptotic series represents a class of asymptotically equivalent functions. The latter property can be applied directly in many cases (see Chapter 1.5).

Asymptotic expansion of functions $f(x)$ and $g(x)$ for $x \to x_0$ regarding the sequence $\{\varphi_n(x)\}$ follows

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \quad g(x) \sim \sum_{n=0}^{\infty} b_n \varphi_n(x),$$

and the following property holds

$$\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) \varphi_n(x).$$

In general, a direct multiplication of the series $\{\varphi_n(x) \cdot \varphi_m(x)\}$ $(m, n = 0, 1, \dots)$ is not allowed, since they sometimes cannot be ordered into an asymptotic sequence. However, it can be done, for instance, in the case $\varphi_n(x) = x^n$. Power series allow division if $b_0 \neq 0$.

Finding logarithms is generally allowed. For instance, let us consider the function $f(x) = (\sqrt{x} \ln x + 2x)e^x$, for which the following relation holds

$$f(x) = [2x + o(x)]e^x \qquad \text{for} \qquad x \to \infty. \tag{1.11}$$

Let $g(x) = \ln[f(x)]$, then according to (1.11), we have

$$g(x) = x + \ln[2x + o(x)] = x + \ln x + \ln 2 + o(1) \sim x + o(x) \quad \text{for} \quad x \to \infty.$$

Raising $g(x)$ to a power we find $f(x) \sim e^x$ for $x \to \infty$. Note that the multiplier $2x$ is lost. The reason is that the carried out involution in series approximation of $g(x)$ does not include terms $\ln x$ and $\ln 2$ acting on the main term of the asymptotic of $f(x)$, and only the quantities of order $o(1)$ do not change the coefficient, since $\exp\{o(1)\} \sim 1$.

The power form asymptotic series

$$f(x) \sim \sum_{n=2}^{\infty} a_n x^{-n} \qquad \text{for} \qquad x \to \infty,$$

may be integrated step by step. Differentiation of asymptotic series are not allowed in general. For example, the function

$$f(x) = e^{-1/x} \sin(e^{-1/x})$$

possesses the following singular power form series

$$f(x) \sim 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \dots,$$