J-P. Serre

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Translated from the French by G. A. Jones

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Preface

These notes are a record of a course given in Algiers from 10th to 21st May, 1965. Their contents are as follows.

The first two chapters are a summary, without proofs, of the general properties of nilpotent, solvable, and semisimple Lie algebras. These are well-known results, for which the reader can refer to, for example, Chapter I of Bourbaki or my Harvard notes.

The theory of complex semisimple algebras occupies Chapters III and IV. The proofs of the main theorems are essentially complete; however, I have also found it useful to mention some complementary results without proof. These are indicated by an asterisk, and the proofs can be found in Bourbaki, *Groupes et Algèbres de Lie*, Paris, Hermann, 1960–1975, Chapters IV–VIII.

A final chapter shows, without proof, how to pass from Lie algebras to Lie groups (complex—and also compact). It is just an introduction, aimed at guiding the reader towards the topology of Lie groups and the theory of algebraic groups.

I am happy to thank MM. Pierre Gigord and Daniel Lehmann, who wrote up a first draft of these notes, and also Mlle. Françoise Pécha who was responsible for the typing of the manuscript.

Jean-Pierre Serre

Contents

CHAPTER I

Nilpotent Lie Algebras and Solvable Lie Algebras

The Lie algebras considered in this chapter are finite-dimensional algebras over a field k. In Secs. 7 and 8 we assume that k has characteristic 0. The Lie bracket of x and y is denoted by $[x, y]$, and the map $y \mapsto [x, y]$ by ad x.

1. Lower Central Series

Let $\mathfrak{g}$ be a Lie algebra. The *lower central series* of $\mathfrak{g}$ is the descending series $(C^n\mathfrak{g})_{n\geqslant 1}$ of ideals of $\mathfrak{g}$ defined by the formulae

$$C^1\mathfrak{g} = \mathfrak{g}$$

$$C^n\mathfrak{g} = [\mathfrak{g}, C^{n-1}\mathfrak{g}] \qquad \text{if } n \geqslant 2.$$

We have

$$C^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

and

$$[C^n\mathfrak{g}, C^m\mathfrak{g}] \subset C^{n+m}\mathfrak{g}.$$

2. Definition of Nilpotent Lie Algebras

Definition 1. *A Lie algebra $\mathfrak{g}$ is said to be nilpotent if there exists an integer n such that $C^n\mathfrak{g} = 0$.*

More precisely, one says that $\mathfrak{g}$ is nilpotent of class $\leqslant r$ if $C^{r+1}\mathfrak{g} = 0$. For $r = 1$, this means that $[\mathfrak{g}, \mathfrak{g}] = 0$; that is, $\mathfrak{g}$ is abelian.

Proposition 1. *The following conditions are equivalent:*

(i) $\mathfrak{g}$ *is nilpotent of class* $\leqslant r$.
(ii) *For all* $x_0, \ldots, x_r \in \mathfrak{g}$, *we have*

$$[x_0,[x_1,[\ldots,x_r]\ldots]] = (\operatorname{ad} x_0)(\operatorname{ad} x_1)\ldots(\operatorname{ad} x_{r-1})(x_r) = 0.$$

(iii) *There is a descending series of ideals*

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = 0$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ *for* $0 \leqslant i \leqslant r - 1$.

Now recall that the *center* of a Lie algebra $\mathfrak{g}$ is the set of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$. It is an abelian ideal of $\mathfrak{g}$.

Proposition 2. *Let* $\mathfrak{g}$ *be a Lie algebra and let* $\mathfrak{a}$ *be an ideal contained in the center of* $\mathfrak{g}$. *Then:*

$$\mathfrak{g} \textit{ is nilpotent} \Leftrightarrow \mathfrak{g}/\mathfrak{a} \textit{ is nilpotent.}$$

The above two propositions show that the nilpotent Lie algebras are those one can form from abelian algebras by successive "central extensions."

(Warning: an extension of nilpotent Lie algebras is not in general nilpotent.)

3. An Example of a Nilpotent Algebra

Let V be a vector space of finite dimension n. A *flag* $D = (D_i)$ of V is a descending series of vector subspaces

$$V = D_0 \supset D_1 \supset \cdots \supset D_n = 0$$

of V such that $\operatorname{codim} D_i = i$.

Let D be a flag, and let $\mathfrak{n}(D)$ be the Lie subalgebra of $\operatorname{End}(V) = \mathfrak{gl}(V)$ consisting of the elements x such that $x(D_i) \subset D_{i+1}$. One can verify that $\mathfrak{n}(D)$ is a nilpotent Lie algebra of class $n - 1$.

4. Engel's Theorems

Theorem 1. *For a Lie algebra* $\mathfrak{g}$ *to be nilpotent, it is necessary and sufficient for* $\operatorname{ad} x$ *to be nilpotent for each* $x \in \mathfrak{g}$.

(This condition is clearly necessary, cf. Proposition 1.)

Theorem 2. *Let* V *be a finite-dimensional vector space and* $\mathfrak{g}$ *a Lie subalgebra of* $\operatorname{End}(V)$ *consisting of nilpotent endomorphisms. Then:*

(a) $\mathfrak{g}$ *is a nilpotent Lie algebra.*
(b) *There is a flag D of V such that* $\mathfrak{g} \subset \mathfrak{n}(D)$.

We can reformulate the above theorem in terms of $\mathfrak{g}$-modules. To do this, we recall that if $\mathfrak{g}$ is a Lie algebra and V a vector space, then a Lie algebra homomorphism $\phi: \mathfrak{g} \to \mathrm{End}(V)$ is called a $\mathfrak{g}$-module structure on V; one also says that ϕ is a linear representation of $\mathfrak{g}$ on V. An element $v \in V$ is called invariant under $\mathfrak{g}$ (for the given $\mathfrak{g}$-module structure) if $\phi(x)v = 0$ for all $x \in \mathfrak{g}$. (This surprising terminology arises from the fact that, if $k = \mathbf{R}$ or $\mathbf{C}$, and if ϕ is associated with a representation of a connected Lie group G on V, then v is invariant under $\mathfrak{g}$ if and only if it is invariant—this time in the usual sense—under G.)

With this terminology, Theorem 2 gives:

Theorem 2′. *Let* $\phi: \mathfrak{g} \to \mathrm{End}(V)$ *be a linear representation of a Lie algebra* $\mathfrak{g}$ *on a nonzero finite-dimensional vector space V. Suppose that* $\phi(x)$ *is nilpotent for all* $x \in \mathfrak{g}$. *Then there exists an element* $v \neq 0$ *of V which is invariant under* $\mathfrak{g}$.

5. Derived Series

Let $\mathfrak{g}$ be a Lie algebra. The *derived series* of $\mathfrak{g}$ is the descending series $(D^n\mathfrak{g})_{n \geqslant 1}$ of ideals of $\mathfrak{g}$ defined by the formulae

$$D^1\mathfrak{g} = \mathfrak{g}$$

$$D^n\mathfrak{g} = [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}] \qquad \text{if } n \geqslant 2.$$

One usually writes $D\mathfrak{g}$ for $D^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

6. Definition of Solvable Lie Algebras

Definition 2. *A Lie algebra* $\mathfrak{g}$ *is said to be solvable if there exists an integer n such that* $D^n\mathfrak{g} = 0$.

Here again, one says that $\mathfrak{g}$ is solvable of derived length $\leqslant r$ if $D^{r+1}\mathfrak{g} = 0$.

EXAMPLES. 1. Every nilpotent algebra is solvable.

2. Every subalgebra, every quotient, and every extension of solvable algebras is solvable.

3. Let $D = (D_i)$ be a flag of a vector space V, and let $\mathfrak{b}(D)$ be the subalgebra of $\mathrm{End}(V)$ consisting of the $x \in \mathrm{End}(V)$ such that $x(D_i) \subset D_i$ for all i. The algebra $\mathfrak{b}(D)$ (a "Borel algebra") is solvable.

Proposition 3. *The following conditions are equivalent:*

(i) $\mathfrak{g}$ *is solvable of derived length* $\leqslant r$.
(ii) *There is a descending series of ideals of* $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_r = 0$$

such that $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ *for* $0 \leqslant i \leqslant r-1$ *(which amounts to saying that that the quotients* $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ *are abelian).*

Thus one can say that solvable Lie algebras are those obtained from abelian Lie algebras by successive "extensions" (not necessarily central).

7. Lie's Theorem

We assume that k is algebraically closed (and of characteristic zero).

Theorem 3. *Let* ϕ: $\mathfrak{g} \to \mathrm{End}(V)$ *be a finite-dimensional linear representation of a Lie algebra* $\mathfrak{g}$. *If* $\mathfrak{g}$ *is solvable, there is a flag D of V such that* $\phi(\mathfrak{g}) \subset \mathfrak{b}(D)$.

This theorem can be rephrased in the following equivalent forms.

Theorem 3'. *If* $\mathfrak{g}$ *is solvable, the only finite-dimensional* $\mathfrak{g}$-*modules which are simple* (irreducible in the language of representation theory) *are one dimensional.*

Theorem 3''. *Under the hypotheses of Theorem* 3, *if* $V \neq 0$ *there exists an element* $v \neq 0$ *of V which is an eigenvector for every* $\phi(x)$, $x \in \mathfrak{g}$.

The proof of these theorems uses the following lemma.

Lemma. *Let* $\mathfrak{g}$ *be a Lie algebra,* $\mathfrak{h}$ *an ideal of* $\mathfrak{g}$, *and* ϕ: $\mathfrak{g} \to \mathrm{End}(V)$ *a finite-dimensional linear representation of* $\mathfrak{g}$. *Let v be a nonzero element of V and let* λ *be a linear form on* $\mathfrak{h}$ *such that* $\lambda(h)v = \phi(h)v$ *for all* $h \in \mathfrak{h}$. *Then* λ *vanishes on* $[\mathfrak{g}, \mathfrak{h}]$.

8. Cartan's Criterion

It is as follows:

Theorem 4. *Let V be a finite-dimensional vector space and* $\mathfrak{g}$ *a Lie subalgebra of* $\mathrm{End}(V)$. *Then:*

$$\mathfrak{g} \text{ is solvable} \Leftrightarrow \mathrm{Tr}(x \circ y) = 0 \qquad \text{for all } x \in \mathfrak{g},\ y \in [\mathfrak{g}, \mathfrak{g}].$$

(This implication $\Rightarrow$ is an easy corollary of Lie's theorem.)

CHAPTER II

Semisimple Lie Algebras (General Theorems)

In this chapter, the base field k is a field of characteristic zero. The Lie algebras and vector spaces considered have finite dimension over k.

1. Radical and Semisimplicity

Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals of $\mathfrak{g}$, the ideal $\mathfrak{a} + \mathfrak{b}$ is also solvable, being an extension of $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ by $\mathfrak{a}$. Hence there is a largest solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$. It is called the *radical* of $\mathfrak{g}$.

Definition 1. *One says that* $\mathfrak{g}$ *is semisimple if its radical* $\mathfrak{r}$ *is* 0.

This amounts to saying that $\mathfrak{g}$ has no abelian ideals other than 0.

EXAMPLE. If V is a vector space, the subalgebra $\mathfrak{sl}(V)$ of $\mathrm{End}(V)$ consisting of the elements of trace zero is semisimple.

(See Sec. 7 for more examples.)

Theorem 1. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{r}$ *its radical.*

(a) $\mathfrak{g}/\mathfrak{r}$ *is semisimple.*
(b) *There is a Lie subalgebra* $\mathfrak{s}$ *of* $\mathfrak{g}$ *which is a complement for* $\mathfrak{r}$.

If $\mathfrak{s}$ satisfies the condition in (b), the projection $\mathfrak{s} \to \mathfrak{g}/\mathfrak{r}$ is an isomorphism, showing (with the aid of (a)) that $\mathfrak{s}$ is semisimple. Thus $\mathfrak{g}$ is a *semidirect product* of a semisimple algebra and a solvable ideal (a "Levi decomposition").