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THE FINITE ELEMENT METHOD FOR SOLID & STRUCTURAL MECHANICS

O.C. Zienkiewicz, R.L. Taylor & D.D. Fox

The Finite Element Method for Solid and Structural Mechanics

Seventh Edition

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List of Figures

1.1	Finite element mesh for tire analysis: (a) tire cross-section; (b) full mesh.	2
1.2	Temperature contours on a disc brake system (provided by Livermore Software Technology Corporation).	2
1.3	Low-pressure die casting simulation. Temperature map in casting wheel and die components. Image courtesy of ESI Group and CMS.	3
1.4	Problem classes range from toys to full-scale systems. Images courtesy of Dassault Systèmes SIMULIA: (a) radio control toy race car; (b) Beechcraft aircraft.	4
1.5	Domain (Ω) and boundary parts for traction (Γ_u) and displacement (Γ_u).	8
2.1	Isoparametric map for four-node two-dimensional quadrilateral: (a) element in $\boldsymbol{\xi}$ coordinates and (b) element in x coordinates.	24
2.2	Boundary conditions for specified displacements.	35
2.3	Boundary conditions for specified traction.	37
2.4	Normal to surface.	38
2.5	Six-node triangular element for **u** with linear p: (a) displacement nodes and (b) pressure nodes.	46
2.6	Estimation of thermophysical properties in phase change problems. The latent heat effect is approximated by a large capacity over a small temperature interval $2\Delta T$.	49
2.7	(a) Mesh; (b) Time dependent freezing zones.	49
2.8	Reactive sphere. Transient temperature behavior for ignition ($\bar{\delta} = 16$) and nonignition ($\bar{\delta} = 2$) cases: (a) induction time versus Frank–Kamenetskii parameter and temperature profiles; (b) temperature profiles for ignition ($\bar{\delta} = 16$) and nonignition ($\bar{\delta} = 2$) transient behavior of a reactive sphere.	50
2.9	Crash analysis: (a) mesh at $t = 0$ ms; (b) mesh at $t = 20$ ms; (c) mesh at $t = 40$ ms.	52
2.10	Retaining wall subjected to earthquake excitation: comparison of experiment (centrifuge) and calculations [28].	53
3.1	Possibility of multiple solutions.	58
3.2	Newton's method.	60

3.3 The modified Newton method: (a) with initial tangent in increment; (b) with initial problem tangent. 62
3.4 The secant method starting from a K^0 prediction. 63
3.5 Direct (or Picard) iteration. 65
3.6 *Regula falsi* applied to line search: (a) extrapolation; (b) interpolation. 66
3.7 One-dimensional interpretation of the Bergan procedure. 69
3.8 Direct integration procedure. 71
4.1 Spring-dashpot models for linear viscoelasticity: (a) Maxwell element; (b) Kelvin element. 78
4.2 Typical viscoelastic relaxation function. 83
4.3 Standard linear viscoelastic solid: (a) model for standard solid; (b) relaxation function. 84
4.4 Mesh and loads for internal pressure on a thick-walled cylinder: (a) four-node quadrilaterals; (b) nine-node quadrilaterals. 87
4.5 Radial stress for internal pressure on a thick-walled cylinder: (a) mixed model; (b) displacement model. 88
4.6 Uniaxial behavior of materials: (a) nonlinear elastic and plastic behavior; (b) ideal plasticity; (c) strain hardening plasticity. 89
4.7 Yield surface and normality criterion in two-dimensional stress space. 90
4.8 Isotropic yield surfaces in principal stress space: (a) Drucker-Prager and Huber-von Mises; (b) Mohr-Coulomb and Tresca. 103
4.9 Loading and unloading directions in stress space. 108
4.10 A generalized plasticity model describing a very complex path, and comparison with experimental data. Undrained two-way cyclic loading of Nigata sand [68]. (Note that in an undrained soil test the fluid restrains all volumetric strains, and pore pressures develop; see Ref. [69].) 110
4.11 Perforated plane stress tension strip: mesh used and development of plastic zones at loads of 0.55, 0.66, 0.75, 0.84, 0.92, 0.98, and 1.02 times σ_y: (a) T3 triangles; (b) plastic zone spread; (c) Q4 quadrilaterals; (d) Q9 quadrilaterals. 112
4.12 Perforated plane stress tension strip: load deformation for strain hardening case ($H = 225$ kg/mm^2). 113
4.13 Limit load behavior for plane strain perforated strip: (a) displacement (displ.) formulation results; (b) mixed formulation results. 114
4.14 Steel pressure vessel: (a) element subdivision and spread of plastic zones; (b) vertical deflection at point A with increasing pressure. 116
4.15 (a) Elasto-plastic; (b) elasto-viscoplastic; (c) series of elasto-viscoplastic models. 119

4.16 Creep in a pressure vessel: (a) mesh end effective stress contours at start of pressurization; (b) effective stress contours 3 h after pressurization. 122

4.17 Uniaxial, axisymmetric compression between rough plates: (a) mesh and problem; (b) pressure displacement result; (c) plastic flow velocity patterns. 123

4.18 Embankment under action of gravity, relative plastic flow velocities at collapse, and effective shear strain rate contours at collapse: (a) associative behavior; (b) nonassociative (zero volume change) behavior. 124

4.19 Underground power station: gravity and prestressing loads. (a) Elastic stresses; (b) "no-tension" stresses. 125

4.20 Cracking of a reinforced concrete beam (maximum tensile strength 200 lb/in^2). Distribution of stresses at various sections [106]: (a) mesh used; (b) section AA; (c) section BB; (d) section CC. 126

4.21 "Laminar" material: (a) general laminarity; (b) laminar in narrow joint. 128

4.22 Π plane section of Mohr-Coulomb yield surface in principal stress space, with $\phi = 25$ deg (solid line); smooth approximation of Eq. (4.175) (dotted line). 130

4.23 Nonuniqueness: mesh size dependence in extension of a homogeneous bar with a strain softening material. (Peak value of yield stress, σ_y, perturbed in a single element.) (a) Stress σ versus strain ε for material; (b) stress $\bar{\sigma}$ versus average strain $\bar{\varepsilon}(= u/L)$ assuming yielding in a single element of length h. 131

4.24 Illustration of a nonlocal approach (work dissipation in failure is assumed to be constant for all elements): (a) an element in which localization is considered; (b) localization; (c) stress-strain curve showing work dissipated in failure. 133

4.25 A nonlinear heat-generation problem illustrating the possibility of multiple or no solutions depending on the heat-generation parameter $\bar{\delta}$; spontaneous combustion [150]: (a) solution mesh and variation of temperature at point C; (b) two possible temperature distributions for $\bar{\delta} = 0.75$. 135

5.1 Reference and deformed (current) configuration for finite deformation problems. 148

6.1 Volumetric deformation. 183

6.2 Uniaxial stretch. Saint-Venant-Kirchhoff and neo-Hookean material. (M) denotes a modified compressible material. 186

6.3 Uniaxial stretch. Arruda–Boyce model. 188

6.4 Incremental deformation motions and configurations. 196

6.5 Necking of a cylindrical bar: eight-node elements. (a) Finite element model; (b) half-bar by symmetry. 201

6.6 Deformed configuration and contours for necking of bar. (a) First invariant (J_1); (b) second invariant (J_2). 202

6.7 Neck radius versus elongation displacement for a half-bar. 203

6.8 Adaptive refinement applied to the problem of a perforated strip. (a) The geometry of the strip and a very fine mesh are used to obtain an "exact" solution; (b) various stages of refinement aiming to achieve a 5% relative energy norm, error at each load increment (quadratic elements T6/3B/3D were used); (c) local displacement results. 204

6.9 Elongation of elements used to model the nearly one-dimensional behavior and the discontinuity. 206

6.10 Adaptive analysis of plastic flow deformation in a perforated plate. (a) Initial mesh, 273 degrees of freedom; (b) final adapted mesh; (c) displacement of an initially uniform grid embedded in the material. 207

6.11 Failure of a rigid footing on a vertical cut. Ideal von Mises plasticity and quadratic triangles with linear variation for pressure (T6/3C) elements are assumed. (a) Geometrical data; (b) coarse mesh; (c) final adapted mesh; (d) displacements after failure; (e) displacement-load diagrams for adaptive mesh and ideal plasticity ($H = 0$); (f) softening behavior. Coarse mesh and adapted mesh results are with a constant H of -5000 and a variable H starting from -5000 at coarse mesh size. 208

6.12 A $p - \delta$ diagram of elasto-plastic slope aiming at 2.5% error in ultimate load (15% incremental energy error) with use of quadratic triangular elements (T6/3B). Mesh A: $u = 0.0$ (coarse mesh). Mesh B: $u = 0.025$. Mesh C: $u = 0.15$. Mesh D: $u = 0.3$. Mesh E: $u = 0.45$. Mesh F: $u = 0.6$. Mesh G: $u = 0.75$. Mesh H: $u = 0.9$. The last mesh (mesh H, named the "optimal mesh") is used for the solution of the problem from the first load step, without further refinement. 209

6.13 Foundation (eccentric loading); ideal von Mises plasticity. (a) Geometry and boundary conditions; (b) adaptive mesh; (c) deformed mesh using T6/1D elements ($H = 0, \nu = 0.49$). 210

6.14 Iceberg gouging of sea floor near buried oil pipelines. Image courtesy of Dassault Systemes SIMULIA and JP Kenny. 211

7.1 A 2D representative volume element mesh: (a) two material region; (b) RVE—mesh and nodes. 217

7.2 Periodic conditions on 2D RVE: (a) square RVE; (b) hexagonal RVE. 222

7.3 RVE for single fiber composite: (a) full RVE mesh; (b) cutaway showing fiber. 226

7.4 RVE for laminated plate. 228

7.5 RVE deformed shapes for fixed and periodic boundary conditions: (a) fixed displacement boundary; (b) periodic boundary. 228

7.6 Cylindrical bending of plate: (a) large displacement shape at maximum loading; (b) load displacement results for *FE* and FE^2 analyses. 229

7.7 Moment-curvature. Meshes for multiscale finite element solutions: (a) 12×12 unit cell mesh; (b) mesh for unit cell and RVE; (c) mesh for coarse scale. 230

7.8 Moment-curvature. Multiscale response vs. fine-scale finite element solution. 231

8.1 Contact between two bodies: (a) no contact condition; (b) contact state. 236

8.2 Contact by finite elements. 237

8.3 Contact between semicircular disks: node-node solution. (a) Undeformed mesh; (b) deformed mesh; (c) vertical stress contours. 238

8.4 Tied interface for a two region problem. 243

8.5 Node-to-surface contact: (a) contact using element interpolations; (b) contact using "smoothed" interpolations. 246

8.6 Node-to-surface contact: gap and normal definition. 247

8.7 Node-to-surface contact: normal vector description. (a) Normal to master facet; (b) normal to slave facet. 249

8.8 Increment of tangential slip. 259

8.9 Contact between semicircular disks: vertical contours for node-to-surface solution. 266

8.10 Contact between a disk and a block—frictionless solution: (a) initial mesh; (b) deformed mesh; (c) σ_1 stress; (d) σ_2 stress. 267

8.11 Contact between a disk and a block: (a) contact pressure at time 4 and (b) total load. 268

8.12 Contact between a disk and a block: total load for various penalty values. 268

8.13 Configurations for a frictional sliding: (a) initial configuration; (b) position at $t = 2$; (c) position at $t = 4$; (d) position at $t = 6$. 269

8.14 Resultant force history for sliding disk: (a) frictionless; (b) friction $\mu = 0.2$. 270

8.15 Initial and final configurations for a billet. 271

9.1 Shapes for pseudo-rigid and rigid body analysis: (a) ellipsoid; (b) faceted body. 278

9.2 Lagrange multiplier constraint between flexible and rigid bodies: (a) rigid-flexible body; (b) Lagrange multipliers. 283

9.3 Spinning disk constrained by a joint to a rigid arm. 286

9.4 Rigid-flexible model for spinning disk: (a) problem definition. Solutions at time (b) $t = 2.5$ units; (c) $t = 5.0$ units; (d) $t = 7.5$ units; (e) $t = 10.0$ units; (f) $t = 12.5$ units; (g) $t = 15.0$ units; (h) $t = 17.5$ units. 290

9.5	Displacements for rigid-flexible model for spinning disk. Displacement at: (a) revolute; (b) disk rim.	291
9.6	Cantilever with tip mass: (a) $t = 2$ units; (b) $t = 4$ units; (c) $t = 6$ units; (d) $t = 10$ units; (e) $t = 12$ units; (f) $t = 14$ units; (g) $t = 16$ units; (h) $t = 18$ units; (i) $t = 20$ units.	292
9.7	Biofidelic rear impact dummy. Image courtesy of Dassault Systèmes SIMULIA.	292
9.8	Sorting of randomly sized particles. Image courtesy of Dassault Systèmes SIMULIA.	293
10.1	Open and closed balls in $\mathbb{R}^n$.	300
10.2	An open set along with its boundary and closure.	300
10.3	One-dimensional example of a tangent space to the graph of f at a point $\mathbf{x}_0$.	304
10.4	The tangent space at a point of an open set.	304
10.5	The tangent map.	305
10.6	Intrinsic interpretation of the tangent map.	307
10.7	A body in $\mathbb{R}^3$.	307
10.8	Coordinate parameterization of a body.	308
10.9	The tangent basis vectors.	311
10.10	A one-dimensional example of a tangent vector.	312
10.11	A simple body parameterized by the functions $\mathbf{\Phi} : \Omega \to \mathbb{R}^3$ and $\boldsymbol{\phi} : \Gamma \to \mathbb{R}^3$.	312
10.12	A simple surface in $\mathbb{R}^3$.	317
10.13	The transition map $\boldsymbol{\varphi}_2^{-1} \circ \boldsymbol{\varphi}_1 : \Omega_1 \subset \mathbb{R}^2 \to \Omega_2 \subset \mathbb{R}^2$.	317
10.14	Coordinate lines on a simple surface.	319
10.15	A coordinate transformation function $f : \mathcal{V} \subset \mathbb{R}^2 \to \mathcal{U} \subset \mathbb{R}^2$.	320
10.16	Important notation for linear elasticity.	324
10.17	A parameterized body in $\mathbb{R}^3$.	326
10.18	Notation for the reference parameterization of a shell.	332
10.19	The mid-surface basis vectors.	334
10.20	Through the thickness surfaces $\mathcal{S}^\alpha$ defined by holding the coordinate ξ^α constant.	335
10.21	The differential force acting on the surface $\mathcal{S}^1$ with tangent basis vectors and unit normal.	336
10.22	The differential element of moment about the mid-surface acting on the surface $\mathcal{S}^1$.	337
10.23	Through the thickness sections $\mathcal{S}^3$.	338
10.24	A finite rotation and stretch of the director $\mathbf{d}^0$.	347
10.25	Reference geometry of the shell.	367
10.26	Complex shell with stiffener panels. A global parameterization of the type $\mathbf{x} = \boldsymbol{\varphi}^0(\xi^1, \xi^2)$ is not possible.	367

10.27 A four-node quadratic element parameterization of the shell mid-surface $\mathcal{S}_e$. 370

10.28 Notation for the assumed strain field on the standard isoparametric element. 382

10.29 Cylindrical bending of a flat strip. 385

10.30 (a) w : Vertical displacement; (b) θ_x : Rotation; (c) Mx : Bending moment for cylindrical bending of a flat strip: 20 element solution. 386

10.31 Barrel (cylindrical) vault: (a) barrel vault geometry and properties; (b) vertical displacement of center section; (c) longitudinal displacement of support. 16×24 element mesh. 387

10.32 Barrel vault of Fig. 10.31. (a) M_1, transverse; M_2, longitudinal; center-line moments. (b) M_{12}, twisting moment at support. 16×24 element mesh. 388

10.33 Barrel vault of Fig. 10.31. Contours of vertical displacement u_1. 388

10.34 Spherical cap: (a) geometry; (b) 768-element mesh. 389

10.35 Spherical cap: (a) radial (u) and (b) vertical (w) displacements for 768-element mesh. 390

11.1 Illustration of the definition of a transition or overlap map on a differentiable manifold. 394

11.2 Two charts covering the unit circle $\mathrm{S}^1 \subset \mathbb{R}^2$. 395

11.3 A manifold and a local coordinate chart. 396

11.4 Chart $(\mathcal{U}, \chi)$ on the unit circle. 397

11.5 North pole chart in the unit sphere. 398

11.6 Differentiability of a map. 398

11.7 Tangent space to a surface in $\mathbb{R}^3$. 399

11.8 Manifold of diffeomorphisms on $\Omega \subset \mathbb{R}^n$. 401

11.9 Tangent map between two manifolds. 402

11.10 Two parameterizations of the same curve $\mathcal{C} \subset \mathbb{R}^3$. 404

11.11 Velocity vector as a tangent field. 406

11.12 Illustration of a vector field along $\mathcal{C}$. 406

11.13 A curve $\mathcal{C}$ and a moving frame, given by mappings $(\boldsymbol{\varphi}, \boldsymbol{\Lambda}) : I \to \mathbb{R}^3 \times SO(3)$. 407

11.14 Covariant derivative on a hyper-surface. 423

11.15 Geometric interpretation of the exponential map. 425

11.16 Geodesics do not necessarily minimize distance. 428

11.17 Geometric interpretation of the exponential map in $GL(3, \mathbb{R})$. 434

11.18 Geometric interpretation of $\boldsymbol{\Lambda} \in SO(3)$. 440

11.19 Geometric significance of $\exp[\widehat{\boldsymbol{\Theta}}]$: *a rotation about* $\boldsymbol{\Theta} \in \mathbb{R}^3$ of *magnitude* $\|\boldsymbol{\Theta}\| \in \mathbb{R}$. 442

12.1 Simple body model. A *placement* $\mathcal{S} \subset \mathbb{R}^3$ of $\mathcal{P}$ in a configuration $\boldsymbol{\kappa} : \mathcal{P} \to \mathcal{S} \subset \mathbb{R}^3$ is an *open set* $\mathcal{S} = \boldsymbol{\kappa}(\mathcal{P}) \subset \mathbb{R}^3$. 450

12.2 Classical setup of continuum mechanics. We think of $\mathcal{B} = \boldsymbol{\kappa}_0(\mathcal{P})$ (the *reference* placement) and $\mathcal{S}_t = \boldsymbol{\kappa}_t(\mathcal{P})$ (the *current* placement) as *manifolds*. 450
12.3 Geometric setup. Mappings between differentiable manifolds. 452
12.4 The convective coordinate system $\mathbf{X} = \chi_0\left(\xi^1,\xi^2,\xi^3\right)$ and $\mathbf{x} = \chi\left(\xi^1,\xi^2,\xi^3\right)$ for the same $\boldsymbol{\xi} = \left(\xi^1,\xi^2,\xi^3\right) \in \Omega$. 452
12.5 Coordinate curves and base vectors of a chart (χ,Ω). 455
12.6 The setup for stress tensors and traction vectors. 462
13.1 Basic geometric objects. 468
13.2 Two configurations of a rod in Euclidean space. 470
13.3 Interpretation of the strain measure $\boldsymbol{\gamma}_t$. 476
13.4 A rod as a three-dimensional body occupying placements $\mathcal{S}_t \subset \mathbb{R}^3$. 479
13.5 Force element on a cross-sectional area $\mathcal{A}_t$. 481
13.6 Setup for balance laws. 494
13.7 $\chi_0 \equiv$ identity, $D\chi_0 = \mathbf{1}, \boldsymbol{\Lambda}_0 = \mathbf{1}, j_0 = 1, \mathbf{H}_0 \equiv \mathbf{0}$. 494
13.8 Parameterized by reference arc length, $\mathbf{t}_3^0 \equiv \dfrac{\partial \varphi_0}{\partial S}$. 498
13.9 Circular ring geometry and deformed shapes: (a) geometry; (b) deformed shapes. 512
13.10 Circular ring load-displacements. 512
13.11 Cantilever L-beam geometry: (a) cross-section A-A; (b) cross-section B-B. 513
13.12 Cantilever L-beam deformed shapes: (a) cross-section; (b) cross-section B-B. 514
13.13 Cantilever L-beam. Load-displacement at tip: (a) cross-section A-A; (b) cross-section B-B. 514
13.14 Co-linear loaded cantilever beam geometry. 514
13.15 Co-linear loaded cantilever beam. Deformed configuration at four time stages: (a) $t = 0.05$; (b) $t = 0.095$; (c) $t = 0.15$; (d) $t = 0.20$. 515
13.16 Co-linear loaded cantilever beam. Plot of tip displacement components vs. time. 515
14.1 The shell body. 521
14.2 A deformation of the shell. 522
14.3 A three-dimensional parameterization of the shell-like body. 525
14.4 Finite rotations in S^2: the vector $\mathbf{t}^0_{\chi 0}$ is rotated to $\mathbf{t}_\chi$ by the angle $\|\mathbf{t}_\triangle\|$, with axis of rotation along $\boldsymbol{\Theta}$. Here $\boldsymbol{\Theta} := \mathbf{t}^0_{\chi 0} \times \mathbf{t}_\triangle$. 555
14.5 A four-node quadrilateral element parameterization of the shell mid-surface $\mathcal{S}_e$. 563
14.6 Geometric interpretation of the update procedure for the director field. Each nodal director describes a curve on S^2 composed of *arcs of geodesic* (i.e., arcs of great circle). 567

14.7	Geometric update procedure in S^2. Nodal directors, which are updated by means of the exponential map, rotate by the amount $\Delta\boldsymbol{\Lambda}_a^k$.	568
14.8	Notation for the assumed strain field on the standard isoparametric element.	578
14.9	Shell model for L-beam: undeformed and deformed view.	582
14.10	Centerline load-displacement for L-beam. Comparison of shell and beam models. Solid lines: shell; dotted line: beam.	582
14.11	Pinched hemisphere: (a) problem geometry; (b) 4×4 mesh and forces.	583
14.12	Pinched hemisphere. Displacements at force locations: (a) displacement; (b) Pian-Sumihara.	584
14.13	Pinched hemisphere. Displacement contours on deformed configuration: (a) u_1 displacement; (b) u_2 displacement.	584
14.14	Buckling of skin-stringer panel. Image courtesy of Dassault Systemes SIMULIA.	585
14.15	Car crash simulation. Image courtesy of Livermore Software Technology Corporation.	585
A.1	Parent coordinates for a quadrilateral.	598
A.2	Node numbering for four-node and nine-node quadrilateral elements.	599
A.3	Parent coordinates and node order for an eight-node brick element.	600
A.4	Parent coordinates and node order for triangular elements: (a) three-node triangle and area coordinates; (b) six-node triangle.	602
A.5	Node order for linear and quadratic tetrahedron elements.	603

List of Tables

1.1	Index Relation between Tensor and Matrix Forms	11
2.1	Gaussian Guadrature Abscissae and Weights for $\int_{-1}^{1} f(\xi)\, \mathrm{d}\xi = \sum_{j=1}^{n} f(\xi_j) w_j$	27
4.1	Invariant Derivatives for Various Yield Conditions	103
10.1	Equation Count and Unknowns for Linear Shell	341
10.2	Final Equation Count and Unknowns for Linear Shell	344

Preface

The present revision of *The Finite Element Method* was undertaken shortly before the passing in January 2009 of our close friend and co-author Olgierd C. (Olek) Zienkiewicz. His inspiration and guidance has been greatly missed in the intervening years, however, we hope that the essence of his writings is retained in the new work so that current and future scholars can continue to benefit from his insights and many contributions to the field of computational mechanics. The story of his life and works is summarized in *International Journal for Numerical Methods in Engineering*, **80**, 2009, pp. 1–45.

It is 46 years since *The Finite Element Method in Structural and Continuum Mechanics* was first published. This book, which was the first dealing with the finite element method, provided the basis from which many further developments occurred. The expanding research and field of application of finite elements led to the second edition in 1971, the third in 1977, the fourth as two volumes in 1989 and 1991, and the fifth as three volumes in 2000. The size of each of these editions expanded geometrically (from 272 pages in 1967 to the sixth edition of nearly 1800 pages). This was necessary to do justice to a rapidly expanding field of professional application and research. Even so, much filtering of the contents was necessary to keep these editions within reasonable bounds.

In the present edition we have retained the complete works as three separate volumes, each one capable of being used without the others and each one appealing perhaps to a different audience.

The first volume *The Finite Element Method: Its Basis and Fundamentals* is designed to cover quite completely all the steps necessary to solve problems represented by linear differential equations. Applications to problems of elasticity, field problems, and plate and shell structural problems form the primary basis from which the finite element steps are enumerated. After a summary of the basic equations in matrix form, chapters on applications to one- to three-dimensional problems are covered. Two methodologies are presented: weak forms (which may be used for any linear differential equation) and variational theorems which are restricted here to steady-state applications. The basic concepts include interpolation of solution variables, numerical integration to evaluate the final matrices appearing in the finite element approximation, and solution of the resulting matrix equations. Both steady-state and transient problems are covered at an early date to permit the methods to be used throughout the volume. The volume also covers the patch test, treatment of constraints arising from near incompressibility and transverse shear deformations in plates and shells, error estimation, adaptivity, and mesh generation.

In this volume we consider more advanced problems in solid and structural mechanics while in a third volume we consider applications in fluid dynamics. It is our intent that the present volume can be used by investigators familiar with the finite element method at the level presented in the first volume or any other basic textbook on the subject. However, the volume has been prepared such that it can stand alone.

The volume has been organized to cover consecutively two main subject areas. In the first part we consider nonlinear problems in solid mechanics and in the second part nonlinear rod and shell problems in structural mechanics.

In Chapters 1–9 we consider nonlinear problems in solid mechanics. In these chapters the special problem of solving nonlinear equation systems is addressed. We begin by restricting our attention to nonlinear behavior of materials while retaining the assumptions on small strain. This serves as a bridge to more advanced studies later in which geometric effects from large displacements and deformations are presented. Indeed, nonlinear applications are of great importance today and of practical interest in most areas of engineering and physics. By starting our study first using a small strain approach we believe the reader can more easily comprehend the various aspects which need to be understood to master the subject matter. We cover in some detail formulations of material models for viscoelasticity, plasticity, and viscoplasticity which should serve as a basis for applications to other material models. In our study of finite deformation problems we present a series of approaches which may be used to solve problems including extensions for multiscale constitutive models, treatment of constraints such as near incompressibility, and rigid and multibody motions.

In the second part of the volume we consider problems in structural mechanics. This part of the book has been rewritten completely and presents an introduction to the mathematical basis used in many recent publications. The presentation is strongly guided by the works of the late Juan Carlos Simo who also influenced works by the second and third authors.

Chapter 10 presents a self-contained development of linear shell theory, which includes a review of mathematical preliminaries necessary for understanding the structural theory and its finite element implementation. Linear shell theory serves as a model problem for the recent trend toward a strong mathematical grounding of the finite element method; linear shell theory is a problem that embodies many important mechanical, geometrical, and numerical analysis concepts that benefit from this modern mathematical perspective. Rounding out the mathematical framework, a comprehensive subset of differential geometry and calculus on manifolds is given in Chapter 11. This chapter gives sufficient mathematical background for understanding the nonlinear continuum mechanics, nonlinear rod theory, and nonlinear shell theory covered in the subsequent chapters.

Chapter 12 summarizes the basic notation and some fundamental concepts in nonlinear three-dimensional continuum mechanics. This chapter revisits the presentation of geometrically nonlinear problems in Chapter 5 within a geometric framework. Specifically, the chapter presents a curvilinear coordinate vector expression of nonlinear continuum mechanics that forms a common departure point for the nonlinear geometrically exact rod and shell theories of Chapters 13 and 14. The primary goal these chapters is to present geometrically exact models in a way that is optimally suited for numerical implementation. Much of the complexity in rods and shells stems from the nature of the structural analysis (and, hence, is present in linear shell theory) rather than from the nonlinear kinematics or exact geometric treatment of the models. Important details, such as parameterization or the definition of stress resultants, can be isolated from the treatment of large deformation.

The volume concludes with a short chapter on computational methods that describes a companion computer program that can be used to solve several of the problem classes described in this volume.

We emphasize here the fact that all three of our volumes stress the importance of considering the finite element method as a unique and whole basis of approach and that it contains many of the other numerical analysis methods as special cases.

Resources to accompany this book

Complete source code and user manual for program *FEAPpv* may be obtained at no cost from the author's web page: www.ce.berkeley.edu/projects/feap.

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Contents

List of Figures xvii
List of Tables xxvii
Preface xxix

CHAPTER 1 General Problems in Solid Mechanics and Nonlinearity 1

1.1 Introduction 1
1.2 Small deformation solid mechanics problems 5
1.2.1 Strong form of equation: Indicial notation 5
1.2.2 Matrix notation 10
1.2.3 Two-dimensional problems 12
1.3 Variational forms for nonlinear elasticity 14
1.4 Weak forms of governing equations 17
1.4.1 Weak form for equilibrium equation 17
1.5 Concluding remarks 18
References 19

CHAPTER 2 Galerkin Method of Approximation: Irreducible and Mixed Forms 21

2.1 Introduction 21
2.2 Finite element approximation: Galerkin method 21
2.2.1 Displacement approximation 23
2.2.2 Derivatives 24
2.2.3 Strain-displacement equations 25
2.2.4 Weak form 25
2.2.5 Irreducible displacement method 26
2.3 Numerical integration: Quadrature 27
2.3.1 Volume integrals 28
2.3.2 Surface integrals 29
2.4 Nonlinear transient and steady-state problems 30
2.4.1 Explicit Newmark method 30
2.4.2 Implicit Newmark method 31
2.4.3 Generalized midpoint implicit form 33
2.5 Boundary conditions: Nonlinear problems 34
2.5.1 Displacement condition 34
2.5.2 Traction condition 37
2.5.3 Mixed displacement/traction condition 38