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INVARIANT DIFFERENTIAL OPERATORS

VOLUME 1: NONCOMPACT SEMISIMPLE LIE ALGEBRAS AND GROUPS

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Preface

Invariant differential operators play a very important role in the description of physical symmetries – recall, e.g., the examples of Dirac, Maxwell, Klein–Gordon, d'Almbert, and Schrödinger equations. Invariant differential operators played and continue to play important role in applications to conformal field theory. Invariant superdifferential operators were crucial in the derivation of the classification of positive energy unitary irreducible representations of extended conformal supersymmetry first in four dimensions, then in various dimensions. Last, but not least, among our motivations are the mathematical developments in the last 50 years and counting.

Obviously, it is important for the applications in physics to study these operators systematically. A few years ago we have given a canonical procedure for the construction of invariant differential operators. Lately, we have given an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced.

Altogether, over the years we have amassed considerable material which was suitable to be exposed systematically in book form. To achieve portable formats, we decided to split the book in two volumes. In the present first volume, our aim is to introduce and explain our canonical procedure for the construction of invariant differential operators and to explain how they are used on many series of examples. Our objects are noncompact semisimple Lie algebras, and we study in detail a family of those that we call "conformal Lie algebras" since they have properties similar to the classical conformal algebras of Minkowski space-time. Furthermore, we extend our considerations to simple Lie algebras that are called "parabolically related" to the initial family.

The second volume will cover various generalizations of our objects, e.g., the AdS/CFT correspondence, quantum groups, superalgebras, infinite-dimensional (super-)algebras including (super-)Virasoro algebras, and (q-)Schrödinger algebras.

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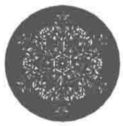
1 Introduction

1.1 Symmetries

The notion of symmetry is a very old one. This is not surprising since there are many natural objects and living beings




which possess symmetry. So since the beginning of civilization people were influenced by this, and by 1200 B.C. symmetry was used extensively in Greek art. These were usually geometric symmetries such as discrete translational symmetry (when some figures were repeated from left to right (or top to bottom)); reflection symmetry with respect to some axis


(combined with translational symmetry); and discrete rotational symmetry (when a figure is not changed upon rotation of a fixed angle).

From the arts the notion of symmetry passed to the sciences. For instance, some symmetrical geometrical figures such as the circle and sphere were considered perfect by the Pythagoreans.

Of course, it was clear that the real world is not exactly symmetric – e.g., take the human body as a nonexact symmetry.

The first appearances of *symmetry in physics* were of geometric nature. It was natural to think that the fundamental constituents of nature should possess some of these symmetries. Indeed, this is the case for many crystals and molecules, which in many cases are symmetrically arranged with respect to reflections as well as discrete translations and rotations. To this day, the study of such discrete symmetries is an interesting field of science.

The use of symmetries in mathematics and physics was enhanced when it was fully realized that symmetries can be described mathematically by expressing a set of *transformations* that leave a particular structure unchanged. This was especially important for the use of *continuous symmetries*.

Thus the set of transformations which leaves the sphere unchanged is the set of rotations of arbitrary angle around the three axes in a three-dimensional Euclidean space.

Mathematically, this is expressed as follows. The sphere of radius r ($\neq 0$) with the center at the beginning of the coordinate system is described as the points with coordinates x_1, x_2, x_3 so that

$$x_1^2 + x_2^2 + x_3^2 = r^2,$$

which can be written in matrix form as

$$\begin{aligned}\phi(x_1, x_2, x_3) &\doteq (x_1, x_2, x_3)\,(x_1, x_2, x_3)^t \\ &= (x_1, x_2, x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= x_1^2 + x_2^2 + x_3^2\end{aligned}$$

and the fact that the rotations are preserving the sphere may be expressed as

$$\phi(x_1', x_2', x_3') = (x_1, x_2, x_3)\, M(\hat{\varphi})\, M(\hat{\varphi})^t\, (x_1, x_2, x_3)^t = \phi(x_1, x_2, x_3),$$

$$M(\hat{\varphi})\, M(\hat{\varphi})^t = I_3 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the 3×3 matrices $M(\hat{\varphi})$ depend on the three angles of rotation in the three possible planes in three dimensions, which is symbolically denoted by $\hat{\varphi}$.

Using so-called Euler angles $\varphi_1, \varphi_2, \vartheta$, the explicit dependence on the rotation angles is shown as follows:

$$M(\hat{\varphi}) = M_1(\varphi_1)\, M_2(\vartheta)\, M_1(\varphi_2),$$

$$M_1(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2(\vartheta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix},$$

where M_1 and M_2 are rotations in the planes (x_1, x_2) and (x_2, x_3), respectively, while the rotations in the plane (x_3, x_1) are given by

$$M_3(\varphi) = M_1(\pi/2)\, M_2(\varphi)\, M_1(3\pi/2) = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{pmatrix}.$$

We note now the properties

$$
\begin{aligned}
M_k(\varphi)\, M_k(\varphi') &= M_k(\varphi + \varphi'), \\
M_k(\varphi)^t &= M_k(-\varphi), \\
M_k(\varphi)\, M_k(\varphi)^t &= M_k(\varphi)^t\, M_k(\varphi) = M_k(0) = I_3.
\end{aligned}
$$

The above properties may be expressed by the mathematical statement that the rotations form a group of transformations: A *group* G is a set of objects $a, b, \ldots,$ (e.g., transformations), for which there exists a rule by which each pair (a, b) of objects in G corresponds again to an object, say c in G, which is called the *product* of a, b, and we simply write $c = ab$. In the example above the product is the product of matrices. Then there is a special object e, called *unit element* (above I_3), such that for every a in G we have $ae = ea = a$. Finally, for each element a there exists another element, called the *inverse* of a and denoted by a^{-1}, such that $aa^{-1} = a^{-1}a = e$; above we have $M(\hat{\varphi})^{-1} = M(\hat{\varphi})^t$. Note that in general $ab \neq ba$. Such groups, for which $ab = ba$ for any choice of a and b, are called *Abelian* groups, otherwise a group is called *non-Abelian*. The group of rotations is not Abelian, e.g.,

$$
M_1(\varphi)\, M_2(\vartheta) \neq M_2(\vartheta)\, M_1(\varphi),
$$

as easily seen from Euler's parametrization.

The rotation group in three-dimensional Euclidean space is denoted in the literature by $SO(3)$. Analogously is defined the group of rotations $SO(n)$ in n-dimensional Euclidean space. From these only $SO(2)$ is Abelian – cf. the matrices M_k for the rotations in a fixed plane. The fact that the rotations in a fixed plane form by itself a group is expressed by saying that $SO(2)$ is a *subgroup* of $SO(3)$. Clearly, if $m < n$ then $SO(m)$ is a subgroup of $SO(n)$.

The groups of rotations are special in another respect. They are *Lie groups*. In general, this means that the elements of the group may be parametrized (the angles above) so that it would become an analytic manifold (real analytic here); moreover, the inverse element correspondence is an analytic function. More importantly for the exposition here is the fact that intimately related to the notion of a Lie group is the notion of a *Lie algebra*.

In general, the Lie algebra $\mathcal{G}$ of the Lie group G is first of all a linear space (over some field of numbers F, here $F = \mathbb{R}$ – the real numbers, and below we shall also use $F = \mathbb{C}$ – the complex numbers) of dimension equal to the dimension of G. It may be identified with the tangent space of G at the unit element of G. Thus, for the group $SO(3)$ the basis of this linear space may be represented as

$$
T_k \doteq \left(\frac{\partial}{\partial \varphi}\, M_k(\varphi)\right)\Big|_{\varphi = 0}
$$

or explicitly:

$$T_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The elements T_k are also called (infinitesimal) generators of the group.

As in every algebra, a Lie algebra has also a product between its elements. For Lie algebras it is a special one called *Lie bracket*. Let $X, Y, \dots$ be elements of a Lie algebra $\mathcal{G}$, then $[X, Y]$ denotes the Lie bracket of X, Y. It has the following special properties which are characteristic of a Lie algebra (over $F = \mathbb{R}, \mathbb{C}$):

$$[X, Y] = -[Y, X]$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Properties above are called *anticommutativity* and *Jacobi identity*, respectively.

In our situation the basis elements T_k are matrices, i.e., we have the ordinary associative product of matrices (for which we do not write the $\cdot$), as well as the commutators. Let us calculate the commutator of, e.g., T_1 and T_2. This is a simple calculation which gives

$$[T_1, T_2] = T_1 T_2 - T_2 T_1 = T_3.$$

Analogously, one obtains

$$[T_j, T_k] = \varepsilon_{jk\ell} T_\ell, \quad j, k, \ell = 1, 2, 3,$$

where $\varepsilon_{jk\ell}$ is totally antisymmetric and $\varepsilon_{123} = 1$.

The Lie algebra of the group $SO(n)$ is denoted by $so(n)$. One may consider the same basis elements as generators of a Lie algebra over the complex numbers $\mathbb{C}$. Then the analogs of $so(n)$ are denoted by $so(n, \mathbb{C})$. The notion of a *subalgebra* is analogous to the subgroup notion, e.g., $so(m)$, resp., $so(m, \mathbb{C})$, is subalgebra of $so(n)$, resp., $so(n, \mathbb{C})$, if $m < n$.

Now for the algebra $so(3, \mathbb{C})$ one may introduce the following basis:

$$X^{\pm} \equiv -iT_1 \mp T_2, \qquad H \equiv -2iT_3.$$

These generators have the following commutators:

$$[H, X^{\pm}] = \pm 2X^{\pm}, \qquad [X^+, X^-] = H.$$