I. P. NATANSON

CONSTRUCTIVE FUNCTION THEORY

VOLUME II

APPROXIMATION IN MEAN

Translated by

JOHN R. SCHULENBERGER

A volume, equally elegant in argumentation and in presentation, of one of the most interesting mathematical branches, dealing with approximate representation of arbitrary functions.

I. P. NATANSON

CONSTRUCTIVE FUNCTION THEORY

Volume II

APPROXIMATION IN MEAN

Translated by

JOHN R. SCHULENBERGER

FREDERICK UNGAR PUBLISHING CO.

NEW YORK

Printed in the United States of America

Library of Congress Catalog Card No. 64—15689

CONTENTS

CONSTRUCTIVE FUNCTION THEORY

APPROXIMATION IN MEAN

I. P. NATANSON

CONSTRUCTIVE FUNCTION THEORY

Volume II

APPROXIMATION IN MEAN

Translated by

JOHN R. SCHULENBERGER

FREDERICK UNGAR PUBLISHING CO.
NEW YORK

Printed in the United States of America

Library of Congress Catalog Card No. 64—15689

CONTENTS

CHAPTER I

THE SPACE $L^2_{p(x)}$

§1. The Problem

In this part of the book, as in the first part, we shall be mainly concerned with the approximation of an arbitrary function by means of polynomials. However, a different criterion of accuracy will now be used.

Up until now we have considered the polynomial approximation of a function continuous in the interval $[a, b]$ to be better the smaller the quantity

$$\max_{a \leqq x \leqq b} |P(x) - f(x)|. \tag{1}$$

We will now call a polynomial $P(x)$ a good approximation to the function $f(x)$ if the value of the integral [1]

$$\int_a^b [P(x) - f(x)]^2 \, dx \tag{2}$$

is small.

It is to be noted that the requirement of continuity of $f(x)$ is a natural one when we use criterion of accuracy (1). For when we approximate a function $f(x)$ arbitrarily close by means of polynomials according to this method, this means that $f(x)$ is the limit function of a *uniformly* convergent sequence of polynomials and is hence itself continuous.

The matter is quite different when the approximation criterion involving the integral (2) is used. For example, suppose that in the interval $[-1, +1]$ the function $f(x)$ is given as follows:

$$f(x) = \begin{cases} 0 \text{ for } & -1 \leqq x \leqq 0, \\ 1 \text{ for } & 0 < x \leqq 1. \end{cases}$$

Although this function is discontinuous at $x = 0$, there exist polynomials $P(x)$ for which

$$\int_{-1}^{+1} [P(x) - f(x)]^2 \, dx$$

becomes arbitrarily small.

[1] We shall ordinarily use somewhat more general integrals than (2) as a criterion of approximation of $P(x)$ to $f(x)$; however, in order not to complicate matters from the outset, we will in the beginning adopt the criterion given in the text.

In order to show this, we define a function $\varphi(x)$ as follows:

$$\varphi(x) = \begin{cases} 0 \text{ for } -1 \leqq x \leqq 0 \\ 1 \text{ for } \dfrac{1}{n} \leqq x \leqq 1, \end{cases}$$

and in the interval $\left[0, \frac{1}{n}\right]$ $\varphi(x)$ increases linearly from 0 to 1. It is obvious that $\varphi(x)$ is continuous in $[-1, +1]$, and moreover

$$\int_{-1}^{+1} [\varphi(x) - f(x)]^2\, dx = \int_{0}^{1/n} [\varphi(x) - f(x)]^2\, dx < \frac{1}{n},$$

since $|\varphi(x) - f(x)| \leqq 1$. On the basis of the WEIERSTRASS theorem, there exists a polynomial $P(x)$ which satisfies the inequality

$$|P(x) - \varphi(x)| < \frac{1}{\sqrt{2n}}$$

for all x in the interval $[-1, +1]$. For this polynomial it follows that

$$\int_{-1}^{+1} [P(x) - \varphi(x)]^2\, dx < \frac{1}{n}.$$

From

$$(a + b)^2 \leqq 2(a^2 + b^2)$$

it follows that

$$[P(x) - f(x)]^2 \leqq 2\{[P(x) - \varphi(x)]^2 + [\varphi(x) - f(x)]^2\},$$

and therefore

$$\int_{-1}^{+1} [P(x) - f(x)]^2\, dx < \frac{4}{n},$$

whence our assertion follows, since n may be chosen arbitrarily large.

Therefore, in our new approach the requirement of continuity of the function to be approximated is unnecessary. We shall therefore no longer impose this restriction, but rather admit discontinuous functions as well. However, these functions may by no means be entirely arbitrary; they must rather be compatible with the existence of the integral (2). If we were to adopt the RIEMANN definition of the integral, this would mean a restriction to those discontinuous functions whose points of discontinuity form a set of measure zero. Such a restriction however would not be intrinsic in the theory now to be developed, but would rather stem from the integral concept employed.

Therefore, in order to achieve the most complete presentation possible, we adopt not the RIEMANN concept of the integral, but rather the LEBESGUE concept. This, to be sure, requires that the reader be familiar with the foundations of the theory of functions of a real variable, which we shall presuppose in what follows.

§2. The Weight Function. The Space $L^2_{p(x)}$

Let a non-negative summable function $p(x)$ be given on the interval $[a, b]$. In view of the special role which this function will play in what follows, we will call it a *weight function* or briefly—a *weight.* We agree once and for all to consider only those weights $p(x)$ which assume the value zero only on a set of measure zero; we shall henceforth make no special mention of this stipulation.

Each weight function $p(x)$ determines two classes of measurable functions on $[a, b]$: the class $L_{p(x)}$ of those functions $f(x)$ for which the product $p(x)f(x)$ is summable and the class $L^2_{p(x)}$ of those functions $f(x)$ for which the product $p(x)f^2(x)$ is summable. If $p(x) = 1$, we designate these classes simply by L and L^2. If in designating these classes it is necessary to indicate the interval of integration as well, we write $L_{p(x)}([a, b])$, $L^2_{p(x)}([a, b])$, $L([a, b])$, and $L^2([a, b])$.

As is shown by the inequality

$$|f(x)| \leqq \frac{1 + f^2(x)}{2},$$

$L^2_{p(x)}$ is contained in $L_{p(x)}$.

The inequality

$$|f(x)g(x)| \leqq \frac{f^2(x) + g^2(x)}{2}$$

further shows that the product of two functions in $L^2_{p(x)}$ belongs to $L_{p(x)}$. This together with the identity

$$(f \pm g)^2 = f^2 \pm 2fg + g^2$$

implies that the sum and difference of two functions in $L^2_{p(x)}$ also belong to $L^2_{p(x)}$. Finally, it is fundamental that with $f(x)$ all functions $cf(x)$, where c is a constant, belong to $L^2_{p(x)}$.

Theorem 1. *If $f(x) \in L^2_{p(x)}$ and $g(x) \in L^2_{p(x)}$, then the following two inequalities, which we shall call the Schwarz inequalities, hold:*

$$\left[\int_a^b p(x)f(x)g(x)\,dx\right]^2 \leqq \left[\int_a^b p(x)f^2(x)\,dx\right]\left[\int_a^b p(x)g^2(x)\,dx\right] \tag{3}$$

$$\sqrt{\int_a^b p(x)[f(x) + g(x)]^2\, dx} \leqq \sqrt{\int_a^b p(x)f^2(x)\, dx} + \sqrt{\int_a^b p(x)g^2(x)\, dx}. \tag{4}$$

Proof. To prove inequality (3) we define

$$\psi(z) = \int_a^b p(x)[zf(x) + g(x)]^2\, dx = Az^2 + 2Bz + C,$$

where

$$A = \int_a^b p(x)f^2(x)\, dx, \quad B = \int_a^b p(x)f(x)g(x)\, dx, \quad C = \int_a^b p(x)g^2(x)\, dx.$$

If $A = 0$, then $f(x) = 0$ (as is usual in the measure theory of functions, we do not distinguish between two functions which differ only on a set of measure zero), and inequality (3) becomes the equality $0 = 0$. If $A > 0$, then (3) is obtained by observing that $\psi(z) \geqq 0$ and

$$\psi\left(-\frac{B}{A}\right) = \frac{AC - B^2}{A}.$$

This completes the proof of (3). We now write (3) in the form

$$\int_a^b pfg\, dx \leqq \sqrt{\int_a^b pf^2\, dx}\sqrt{\int_a^b pg^2\, dx},$$

multiply by 2, and add to both sides the expression

$$\int_a^b pf^2\, dx + \int_a^b pg^2\, dx;$$

we hereby obtain the inequality

$$\int_a^b p(f + g)^2\, dx \leqq \left[\sqrt{\int_a^b pf^2 dx} + \sqrt{\int_a^b pg^2\, dx}\right]^2,$$

which is equivalent to (4).

With each function $f(x) \in L^2_{p(x)}$ we now associate the number

$$\|f\| = \sqrt{\int_a^b p(x)f^2(x)\, dx},$$

which we call the *norm* of the function $f(x)$; the norm has properties similar to those of the absolute value of a number:

I. $\|f\| \geqq 0$, *and* $\|f\| = 0$ *if and only if* $f = 0$.

II. $\|cf\| = |c| \cdot \|f\|$, *and in particular* $\|-f\| = \|f\|$.

III. $\|f + g\| \leqq \|f\| + \|g\|$.

This norm concept makes it possible for us to introduce a useful geometric interpretation.

Let E be a set of elements $x, y, z, \ldots$ of any type whatsoever.

With each pair of elements x, y let a real number $r(x, y)$ be associated with the properties

1. $r(x, y) \geqq 0$, *and* $r(x, y) = 0$ *if and only if* $x = y$.
2. $r(x, y) = r(y, x)$.
3. $r(x, z) \leqq r(x, y) + r(y, z)$.

When all these conditions are met, the set E is said to form a *metric space, and* $r(x, y)$ *is called the distance between* x *and* y.

If we define the distance between two functions f and g in $L^2_{p(x)}$ as

$$r(f, g) = \|f - g\|,$$

then $L^2_{p(x)}$ becomes a metric space.

§3. Convergence in Mean

Definition 1. An element f of the space $L^2_{p(x)}$ is called a *limit* of a sequence $f_1, f_2, f_3, \ldots, f_n, \ldots$ of elements of the same space if

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

We will usually abbreviate this relation

$$\lim_{n\to\infty} f_n = f, f_n \to f,$$

although its function-theoretic meaning is actually

$$\lim_{n\to\infty} \int_a^b p(x)[f_n(x) - f(x)]^2\, dx = 0.$$

Convergence of this type we call *convergence in mean with the weight function* $p(x)$.

Theorem 1. *A sequence of elements of* $L^2_{p(x)}$ *cannot have two distinct limits.*

Proof. Suppose $f_n \to f$ and $f_n \to g$. Passing to the limit in the inequality

$$0 \leqq \|f - g\| \leqq \|f - f_n\| + \|f_n - g\|$$

gives the relation $\|f - g\| = 0$, and hence $f = g$.

Theorem 2. *If a sequence of functions* $\{f_n(x)\}$ *converges in mean to the function* $f(x)$, *then it is possible to select from it a subsequence* $\{f_{n_k}\}$ *which converges to* $f(x)$ *almost everywhere.*

The proof of this theorem is based on the following important theorem from function theory (which we do not prove here):

Theorem (B. LEVI).[2] *Given a sequence of non-negative measurable functions*

$$u_1(x), \quad u_2(x), \quad u_3(x), \quad \ldots$$

on the interval $[a, b]$ *such that*

$$\sum_{k=1}^{\infty} \int_a^b u_k(x)\, dx < +\infty,$$

then

$$\lim_{n \to \infty} u_n(x) = 0$$

almost everywhere in $[a, b]$.

Proof of Theorem 2. Choose a sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that

$$\int_a^b p(x)[f_{n_k}(x) - f(x)]^2\, dx < \frac{1}{k!}.$$

Then from the LEVI theorem

$$\lim_{k \to \infty} p(x)[f_{n_k}(x) - f(x)]^2 = 0$$

almost everywhere in $[a, b]$, and since $p(x)$ is positive almost everywhere, the proof of the theorem is completed.

Theorem 3. *If $f_n \to f$, then $\|f_n\| \to \|f\|$.*

Proof. From

$$\|f\| \leqq \|f_n\| + \|f - f_n\|, \quad \|f_n\| \leqq \|f\| + \|f_n - f\|$$

one obtains

$$|\, \|f_n\| - \|f\|\, | \leqq \|f_n - f\|,$$

whence the assertion follows.

Definition 2. A sequence $\{f_n\} \in L^2_{p(x)}$ is called *convergent in itself* if to every $\varepsilon > 0$ there corresponds an integer N such that $n > N$ and $m > N$ imply

$$\|f_n - f_m\| < \varepsilon. \tag{5}$$

[2] B. LEVI stated the theorem in a more general context. Cf. I. P. Natanson [5], p. 141.