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Algebraic Spaces and Stacks

Martin Olsson

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Algebraic Spaces and Stacks

To Jasmine

Preface

The theory of algebraic spaces and stacks has its origins in the study of moduli spaces in algebraic geometry. It is closely related to the problem of constructing quotients of varieties by equivalence relations or group actions, and the basic definitions are natural outgrowths of this point of view. The foundations of the theory were introduced by Deligne and Mumford in their fundamental paper on the moduli space of curves [**23**] and by Artin building on his work on algebraic approximations [**9**]. Though it has taken some time, algebraic spaces and stacks are now a standard part of the modern algebraic geometers toolkit and are used throughout the subject.

This book is an introduction to algebraic spaces and stacks intended for a reader familiar with basic algebraic geometry (for example Hartshorne's book [**41**]). We do not strive for an exhaustive treatment. Rather we aim to give the reader enough of the theory to pursue research in areas that use algebraic spaces and stacks, and to proceed on to more advanced topics through other sources. Numerous exercises are included at the end of each chapter, ranging from routine verifications to more challenging further developments of the theory.

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Introduction

A basic theme in algebraic geometry is to classify various geometric structures using moduli spaces or parameter spaces, and the study of such classification problems leads naturally to the notions of algebraic spaces and stacks. Just as nonreduced or nonseparated schemes are important even if one is only motivated by the study of smooth projective varieties, so algebraic spaces and stacks are natural extensions of the notion of scheme.

An important classical situation where one encounters algebraic spaces and stacks is when constructing quotients of schemes by group actions. If X is a scheme and G is a finite group acting freely on X then the quotient X/G of X by the G-action always exists as an algebraic space, but this quotient may not exist as a scheme (see 5.3.2 for an example). More generally, one can construct algebraic spaces as quotients of schemes by free actions of algebraic groups. Loosely speaking, an algebraic space is a geometric object obtained by gluing together schemes using the étale topology rather than the Zariski topology. An algebraic space can always be realized as the quotient of a scheme X by an equivalence relation $\Gamma \hookrightarrow X \times X$ such that the two projections $\Gamma \to X$ are étale. Of course a given algebraic space can have many different presentations as a quotient, and care has to be taken in defining the proper category in which the quotient can be taken.

Algebraic stacks enter into the study of group actions on schemes when the action is no longer free. If G is a group, say finite for simplicity, acting on a scheme X the equivalence relation $\Gamma \subset X \times X$ given by declaring points in the same orbit equivalent, need no longer be an étale equivalence relation. In topology it is well-known how to form quotients by nonfree actions. Namely, one considers a contractible space EG with free G-action and then defines the quotient $[X/G]$ to be the quotient of $X \times EG$ by the diagonal action. This quotient has many advantages compared to the naive quotient. For example, the projection map $X \times EG \to [X/G]$ is a principal G-bundle. In algebraic geometry there is no analogue of the space EG, but still one can form the stack quotient $[X/G]$ for a group acting on a scheme X and this quotient enjoys many of the good properties of the corresponding quotient in topology. This stack quotient $[X/G]$ is no longer a space but something more general. For instance, if k is an algebraically closed field then the k-points of the stack is not a set but rather a category; namely, the category whose objects are the k-points of X and for which a morphism $x \to x'$ is given by an element $g \in G$ such that $gx = x'$.

Not all algebraic stacks are quotients of schemes by group actions, though many important algebraic stacks can be realized in this way. To get a sense of the general definition, recall that the Yoneda imbedding identifies the category of schemes with a subcategory of the category of functors

$$(\text{schemes})^{\text{op}} \to (\text{Sets}).$$

Under this identification a scheme X corresponds to the functor h_X sending a scheme T to the set $\text{Hom}(T, X)$ of morphisms of schemes $T \to X$. A functor is called *representable* if it is isomorphic to h_X for some scheme X. From this point of view, algebraic stacks can be viewed as the natural generalization of schemes obtained by replacing the category of sets by groupoids (categories in which all morphisms are isomorphisms). So an algebraic stack can be viewed as a functor

$$(\text{schemes})^{\text{op}} \to (\text{Groupoids})$$

satisfying certain conditions generalizing those characterizing schemes among all functors from schemes to sets. To make this precise one encounters a number of technical difficulties including the fact that groupoids do not form a category but a 2-category (functors between categories do not form a set but rather a category). Once these are overcome, however, one can generalize most of standard algebraic geometry to algebraic stacks. This includes the theory of quasi-coherent and coherent sheaves, cohomology (both cohomology of quasi-coherent sheaves as well as étale and other theories), intersection theory and so on. In fact, the more general setting of algebraic stacks also allows some constructions that are impossible in the ordinary category of schemes (see for example the root stack construction in section 10.3).

Most functors

$$M : (\text{schemes})^{\text{op}} \to (\text{Sets})$$

occurring in moduli theory can naturally be lifted to functors (ignoring the technical issues just mentioned)

$$\mathscr{M} : (\text{schemes})^{\text{op}} \to (\text{Groupoids}).$$

Typically the functor M sends a scheme T to the set of isomorphism classes of certain geometric objects over T (for example families of curves of a given genus g, vector bundles on a fixed variety pulled back to T, etc.). The lifting $\mathscr{M}$ is obtained by sending T to the groupoid whose objects are the geometric objects in question over T and whose morphisms are isomorphisms between them. In this way M is obtained from $\mathscr{M}$ by sending a scheme T to the set of isomorphism classes in the groupoid $\mathscr{M}(T)$.

A *fine moduli space* for the moduli problem is a scheme X and an isomorphism of functors $h_X \simeq M$. As we illustrate below, for many moduli problems the presence of nontrivial automorphisms prevents the moduli problem from having a fine moduli space. In this situation one classically looks for a *coarse moduli space*, which is a scheme X with a morphism of functors $M \to h_X$ that is universal for morphisms from M to representable functors and such that for any algebraically closed field k the induced map $M(\text{Spec}(k)) \to h_X(\text{Spec}(k)) = X(k)$ is a bijection. Thus, for example, if we wish to classify curves of genus g over a field k a coarse moduli space X would, in particular, have its points in 1-1 correspondence with isomorphism classes of such curves. It would also be a fine moduli space if there exists a family of curves $C_X \to X$ over X such that for any k-scheme T and family of curves Y/T of genus g there exists a unique morphism $T \to X$ such that Y is the pullback of C_X. In this case we say also "X classifies families of curves of genus g."

This definition also gives a definition of coarse moduli space for stacks: A coarse moduli space for a stack $\mathscr{M}$ is a coarse moduli space for the corresponding functor of isomorphism classes. As we discuss in this book, the Keel-Mori theorem gives some natural conditions under which an algebraic stack has a coarse moduli space. The typical situation in moduli theory is that one does not have a fine moduli space but the stack $\mathscr{M}$ is a nice algebraic stack admitting a coarse moduli space. In this situation, the stack $\mathscr{M}$ is often a better object to work with than the coarse moduli space. For example, the algebraic stack $\mathscr{M}$ could be smooth and the coarse space singular; also one has a so-called universal family over $\mathscr{M}$ but not over the coarse space. It is often advantageous to study the geometry of $\mathscr{M}$ directly without passing to the coarse moduli space.

Rather than delving into too many technicalities about the exact definitions of algebraic stacks here in the introduction, let us illustrate some of these ideas with the moduli of elliptic curves. For simplicity we work here over $\mathbb{C}$; a more complete discussion can be found in the main text (in particular Chapter 13, §13.1; see also [**42**, §26] for another point of view). We learned the following example from [**39**, Chapter 2].

Recall (for example from [**41**, Chapter IV, §4]) that an *elliptic curve* over $\mathbb{C}$ is a pair (E, e), where E is a smooth projective genus 1 curve over $\mathbb{C}$, and $e \in E$ is a closed point (we often denote such an elliptic curve simply by E, omitting the marked point from the notation). Any such curve can be described as the zero locus in $\mathbb{P}^2$ of a homogeneous polynomial of the form

$$Y^2Z - X(X-Z)(X-\lambda Z),$$

where $\lambda \in \mathbb{C} - \{0,1\}$ and the point e is given by $[0:1:0]$. This polynomial in fact defines a family

$$\mathscr{E} \hookrightarrow \mathbb{P}^2 \times (\mathbb{A}^1 - \{0,1\}),$$

over the punctured affine line, in which every elliptic curve is isomorphic to a fiber over some point $\lambda \in \mathbb{A}^1 - \{0,1\}$. The line $\mathbb{A}^1 - \{0,1\}$ can therefore be thought of as a parameter space for elliptic curves. However, the representation of an elliptic curve as a fiber is not unique and depends on a choice of equation for the curve as a subscheme of $\mathbb{P}^2$. In fact, there is an action of the symmetric group S_3 on $\mathbb{A}^1 - \{0,1\}$ generated by the automorphisms

$$\lambda \mapsto 1/\lambda, \quad \lambda \mapsto \frac{1}{1-\lambda},$$

and one checks that two points $\lambda, \lambda' \in \mathbb{C} - \{0,1\}$ define isomorphic elliptic curves if and only if they lie in the same S_3-orbit. Therefore, if we want to parametrize abstract elliptic curves without a projective imbedding, we should consider the quotient of $\mathbb{A}^1 - \{0,1\}$ by this S_3-action. In the category of schemes this quotient is given by taking the spectrum of the S_3-invariants in the ring $\mathbb{C}[\lambda]_{\lambda(\lambda-1)}$ which it turns out is isomorphic to the polynomial ring $\mathbb{C}[j]$ in one variable j given by the formula

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

This implies that there is a bijection between isomorphism classes of elliptic curves over $\mathbb{C}$ and complex numbers $j \in \mathbb{C}$. The affine line $\mathbb{A}^1_j$ is not, however, a fine moduli space for moduli of elliptic curves. There does not exist a family of elliptic curves $\mathscr{E} \to \mathbb{A}^1_j$ such that for any $\mathbb{C}$-scheme T and family of elliptic curves E/T

there exists a unique morphism $\rho_E : T \to \mathbb{A}^1_j$ such that E is isomorphic to the pullback along ρ_E of $\mathscr{E}$ (see 13.1.1 for the precise definition of an elliptic curve over a scheme S).

This can be seen explicitly as follows. Consider the family of elliptic curves $\mathscr{E}_t$ defined over the t-line $\mathbb{A}^1_t - \{0\}$ given by the equation

$$Y^2Z = X^3 - tZ^3.$$

The j-invariant of every fiber (including the generic fiber) is 0, so the corresponding map $\mathbb{A}^1_t - \{0\} \to \mathbb{A}^1_j$ is a constant map. The elliptic curve E_0 given by $Y^2Z = X^3 - Z^3$ also has j-invariant 0, so if the j-line was a fine moduli space for elliptic curves one would expect that the family $\mathscr{E}_t$ was isomorphic to $E_0 \times (\mathbb{A}^1_t - \{0\})$. This is not the case, however. Indeed, over the function field $\mathbb{C}(t)$ these two curves are not isomorphic, but they do become isomorphic over the field extension $\mathbb{C}(t^{1/6})$ (this is exercise 4.D in the text; the reader may wish to work out this exercise now).

This example not only shows that the j-line $\mathbb{A}^1_j$ is not a fine moduli space for moduli of elliptic curves, but in fact that the functor

$$M_{1,1} : (\mathbb{C}\text{-schemes})^{\text{op}} \to \text{Sets}$$

sending a scheme S to the set of isomorphism classes of elliptic curves over S is not representable. Indeed, the preceding discussion implies that $M_{1,1}$ fails one of the basic properties of representable functors; so-called *étale descent*. If U is a $\mathbb{C}$-scheme and

$$F : (\mathbb{C}\text{-schemes})^{\text{op}} \to \text{Sets}$$

is the functor sending a scheme S to the set of morphisms $S \to U$, then for any field extension $K \hookrightarrow L$ the map

$$F(\text{Spec}(K)) \to F(\text{Spec}(L))$$

is injective. As exercise 4.D demonstrates, the failure of this condition for $M_{1,1}$ arises from the fact that elliptic curves have nontrivial automorphisms. On the other hand, there is an algebraic stack $\mathscr{M}_{1,1}$ which associates to a scheme S the groupoid whose objects are elliptic curves over S with isomorphisms between them. This stack $\mathscr{M}_{1,1}$ is a so-called Deligne-Mumford stack, it is smooth, there is a universal elliptic curve $\mathscr{E} \to \mathscr{M}_{1,1}$ over it, and so on. Furthermore, the stack $\mathscr{M}_{1,1}$ admits a coarse moduli space which in this case is the natural map $\mathscr{M}_{1,1} \to \mathbb{A}^1_j$ associating to an elliptic curve its j-invariant.

Outline of the book:

A proper treatment of the subject requires a fair amount of foundational work. Our general philosophy is that covering a subset of the theory with all the details included is preferable to a cursory exposition that covers more topics. A considerable portion of the book is therefore devoted to foundational topics such as Grothendieck topologies, descent, and fibered categories. While perhaps a bit 'dry', these subjects are so central to the theory that any student wishing to study the subject needs a thorough understanding of these topics.

Chapter 1 collects together various facts from "standard algebraic geometry" that, nonetheless, may not be part of a standard course on the subject. The reader may skip this chapter referring back as needed later. In addition to technical background, this chapter also contains a discussion of schemes from a more functor-theoretic point of view than the standard treatment. This serves as a lead-in to the definitions of algebraic spaces and stacks, and it may be helpful for the reader to consider this functor-theoretic point of view of schemes first.

As mentioned above, schemes can be viewed via the Yoneda imbedding as functors satisfying various properties with respect to the Zariski topology reflecting the fact that they can be covered by affines. The notion of algebraic space is obtained from this point of view by replacing the Zariski topology by the étale topology. For this to make sense we need some foundational material on Grothendieck topologies and sites, and we develop this material in Chapter 2. In this chapter we also discuss some technical points about simplicial topoi, which provide a basic technical tool (in particular for cohomological descent).

One way to phrase the notion of stack is to say that a stack is a sheaf taking values in groupoids (categories in which all morphisms are isomorphisms), instead of sets. Making this definition precise, however, is nontrivial and one has to tackle several 2-categorical issues. Chapter 3 is devoted to the basic definitions and results concerning fibered categories, which provide a solution to these 2-categorical issues.

Chapter 4 discusses descent, which provides the categorical version of the sheaf condition for presheaves. Of basic importance here is faithfully flat descent, which enables one to work with bigger topologies, such as the fppf or étale topologies, in the algebro-geometric context. In this chapter we also discuss some basic results about torsors and principal homogeneous spaces, which play an important role in many examples, and introduce the condition for a fibered category to be a stack.

We then proceed to the development of the theory of algebraic spaces in Chapter 5. The main difference between our treatment and that in other standard sources (for example Knutson's book [**45**]) is that we make no assumptions on the diagonal in the definition of algebraic space. This we feel is the most natural approach though it entails a little extra care in the development of the theory.

To develop the theory of algebraic spaces beyond the basic definitions we need some results about quotients of schemes by finite flat equivalence relations. We discuss these in Chapter 6. This is a very classical topic which is useful both for algebraic spaces but also for the discussion of coarse moduli spaces in later chapters. In Chapter 6 we show in particular that reasonable algebraic spaces can be stratified by schemes.

In Chapter 7 we turn to the study of quasi-coherent sheaves on algebraic spaces. We discuss Stein factorization, Chow's lemma, and finiteness of cohomology. At this point we have generalized most of the standard scheme theory to algebraic spaces, and now turn our attention to algebraic stacks.

In Chapter 8 we introduce the basic definitions for algebraic stacks, as well as several examples. We also discuss Deligne-Mumford stacks, which have many special properties not enjoyed by general algebraic stacks.

Chapter 9 is devoted to the study of quasi-coherent sheaves on algebraic stacks. We have chosen to work with the lisse-étale site as in [**49**], though there are persuasive arguments for using bigger topologies such as the fppf topology. In particular, working with the lisse-étale site introduces some complications when considering pullbacks of quasi-coherent sheaves, and we discuss how to handle these in Chapter 9.

In Chapter 10 we discuss several basic constructions and examples. The notion of a proper morphism of stacks is introduced, and among other examples we discuss root stacks and a stack-theoretic version of the usual Proj construction.

We then turn to the Keel-Mori theorem and the construction of coarse moduli spaces in Chapter 11. Briefly a coarse moduli space for an algebraic stack $\mathscr{X}$ is a morphism $\pi : \mathscr{X} \to X$ to an algebraic space that is universal for such morphisms and such that for an algebraically closed field k the map π induces a bijection between isomorphism classes of the groupoid $\mathscr{X}(k)$ and $X(k)$. A basic example is the moduli stack $\mathscr{M}_{1,1}$ of elliptic curves discussed above, where the j-invariant defines a map $\mathscr{M}_{1,1} \to \mathbb{A}^1$, which in fact is a coarse moduli space.

The last two chapters discuss applications of the theory.

In Chapter 12 we discuss gerbes and their connection with Azumaya algebras and cohomology classes. A gerbe bound by a smooth abelian group scheme $\boldsymbol{\mu}$ over a scheme X is a special kind of algebraic stack over X. The isomorphism classes of such gerbes are given by $H^2(X, \boldsymbol{\mu})$. For $\boldsymbol{\mu} = \mathbb{G}_m$ we get a bijection between $\mathbb{G}_m$-gerbes over X and the group $H^2(X, \mathbb{G}_m)$, which is closely related to the Brauer group of X. Some questions about Brauer groups can be reformulated to algebro-geometric questions about the associated $\mathbb{G}_m$-gerbes, and we discuss some of this in Chapter 12.

In Chapter 13 we discus various moduli stacks of curves. In some sense this brings the book back to the original paper of Deligne and Mumford [**23**]. We discuss moduli of elliptic curves, the Deligne-Mumford compactification $\overline{\mathscr{M}}_g$ of the moduli stack of curves of genus $g \geq 2$, and moduli of stable maps. In this chapter we use some facts from curve theory that may not be so familiar to the beginning reader (for example, the stable reduction theorem), but we summarize the basic results used. The aim here is to synthesize the theory of stacks with classical curve theory.

Finally, there is a glossary where we summarize some of the category theory used in the text.

Nothing in this book is original, and all the results herein can be found in some form in the existing literature. We indicate our main sources in each chapter introduction, and to help the reader connect with the literature we also provide precise references for the main theorems. The basic treatments for the material presented in this book are [**9**], [**23**], [**36**], [**45**], [**49**], and [**71**]. One major topic that we do not cover in this book is Artin's method for proving representability by an algebraic space or stack using deformation theory. The original papers of Artin [**4**], [**5**], [**6**], [**7**], and [**9**] remain the definitive references on this topic.

CHAPTER 1

Summary of background material

In this chapter we review some of the basic facts and definitions that we will need later. The reader can browse this chapter to start, referring back only when needed.

1.1. Flatness

For a more detailed discussion of the notion of flatness see [**2**, Chapter V].

1.1.1. Let R be a ring. Recall that an R-module M is called *flat* if the functor

$$(-) \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$$

is an exact functor, where Mod_R denotes the abelian category of R-modules. The module M is called *faithfully flat* if M is flat, and if for any two R-modules N and N' the natural map

$$\mathrm{Hom}_R(N, N') \to \mathrm{Hom}_R(N \otimes_R M, N' \otimes_R M)$$

is injective.

PROPOSITION 1.1.2. *Let M be an R-module. The following are equivalent:*

(i) *M is faithfully flat.*

(ii) *M is flat and for any R-module N' the map*

$$N' \to \mathrm{Hom}_R(M, N' \otimes M), \quad y \mapsto (m \mapsto y \otimes m) \tag{1.1.2.1}$$

is injective.

(iii) *A sequence of R-modules*

$$N' \to N \to N'' \tag{1.1.2.2}$$

is exact if and only if the sequence

$$N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M \tag{1.1.2.3}$$

is exact.

(iv) *A morphism of R-modules $N' \to N$ is injective if and only if the morphism $N' \otimes_R M \to N \otimes_R M$ is injective.*

(v) *M is flat and if $N \otimes_R M = 0$ for some R-module N, then $N = 0$.*

(vi) *M is flat and for every maximal ideal $\mathfrak{m} \subset R$ we have $M/\mathfrak{m}M \neq 0$.*

PROOF. First let us show that (i) is equivalent to (ii). If $F \to N$ is a surjective morphism of R-modules, then for any R-module N' we have a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_R(N, N') & \longrightarrow & \mathrm{Hom}_R(N \otimes M, N' \otimes M) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_R(F, N') & \longrightarrow & \mathrm{Hom}_R(F \otimes M, N' \otimes M), \end{array}$$

where the vertical maps are injective. In particular, choosing $F = \bigoplus_{i\in I} R$ to be a free module, in which case we have natural isomorphisms

$$\mathrm{Hom}_R(F, N') \simeq \prod_{i\in I} N', \quad \mathrm{Hom}_R(F\otimes M, N'\otimes M) \simeq \prod_{i\in I} \mathrm{Hom}_R(M, N'\otimes M),$$

we see that an R-module M is faithfully flat if and only if for any R-module N' the map

$$N' \to \mathrm{Hom}_R(M, N'\otimes M), \quad y \mapsto (m \mapsto y\otimes m)$$

is injective, thereby showing the equivalence of (i) and (ii).

Next, let us show that (ii) implies (iv). If M is flat and $N' \to N$ is injective, then $N'\otimes M \to N\otimes M$ is also injective by the flatness assumption on M. So to show that (ii) implies (iv) it suffices to show that if (ii) holds and $N' \to N$ is a morphism of R-modules such that $N'\otimes M \to N\otimes M$ is an injection, then $N' \to N$ is also an inclusion. For this, note that we obtain a commutative diagram

$$(1.1.2.4) \qquad \begin{array}{ccc} N' & \hookrightarrow & \mathrm{Hom}_R(M, N'\otimes M) \\ \downarrow & & \downarrow \\ N & \hookrightarrow & \mathrm{Hom}_R(M, N\otimes M), \end{array}$$

where the horizontal arrows are inclusions by (ii). If $N'\otimes M \to N\otimes M$ is an inclusion, then so is the right vertical arrow in the diagram, from which it follows that $N' \to N$ is also injective.

Statement (v) follows from (iv) applied to the map $N \to 0$.

Statement (v) implies (vi) by taking $N = R/\mathfrak{m}$.

Also (v) is equivalent to (iii). Indeed (iii) is equivalent to the statement that M is flat and for any sequence (1.1.2.2) for which (1.1.2.3) is exact the sequence (1.1.2.2) is exact. Now observe that if M is flat and (1.1.2.2) is a sequence of R-modules then setting

$$H := \mathrm{Ker}(N \to N'')/\mathrm{Im}(N' \to N)$$

we have

$$H\otimes M := \mathrm{Ker}(N\otimes M \to N''\otimes M)/\mathrm{Im}(N'\otimes M \to N\otimes M).$$

From this it follows that (v) and (iii), and the converse direction is immediate (consider the sequence $0 \to N \to 0$).

To prove the proposition we are therefore reduced to showing that (vi) implies (ii). For this we show that a counterexample to (ii) yields a counterexample for (vi). So suppose N' is an R-module and $x \in N'$ is a nonzero element mapping to zero under the map (1.1.2.1). Denote by $L \subset N'$ the submodule generated by x, and let $\mathfrak{a} \subset R$ be the kernel of the map $R \to N'$ sending $f \in R$ to $f\cdot x$, so we have $R/\mathfrak{a} \simeq L$. If M is flat, then the map

$$L\otimes M \to N'\otimes M$$

is an inclusion, from which we deduce that the map (1.1.2.1) for L,

$$L \to \mathrm{Hom}_R(M, L\otimes M),$$

is the zero map. The image of $x \in L$ under this map is via the isomorphism $L \simeq R/\mathfrak{a}$ identified with the projection map

$$M \to M\otimes R/\mathfrak{a} \simeq M/\mathfrak{a}M,$$