Probability and Randomness

Quantum versus Classical

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To my son Anton

Preface

The education system for physics students worldwide suffers from the absence of a deep course in probability and randomness. This is a real problem for students interested in quantum information theory, quantum optics, and quantum foundations. Here a primitive treatment of probability and randomness may lead to deep misunderstanding of the theory and wrong interpretations of experimental results. Since my visits (in 2013 and 2014 by kind invitations of C. Brukner and A. Zeilinger) to the Institute for Quantum Optics and Quantum Information (IQOQI) of Austrian Academy of Sciences, a number of students (experimentalists!) have been asking me about foundational problems of probability and randomness, especially inter-relation between classical and quantum structures. I gave two lectures on these problems [165]. Surprisingly, experiment-oriented students demonstrated very high interest in mathematical peculiarities. This (as well as frequent reminder of Prof. Zeilinger) motivated me to write a text based on these lectures which were originally presented in the traditional black-board form. The main aim of this book is to provide a short foundational introduction to classical and quantum probability and randomness.

Chapter 1 starts with the presentation of the Kolmogorov (1933) *measure-theoretic axiomatics.* The von Mises frequency probability theory which preceded the Kolmogorov theory is also briefly presented.[1] In this chapter we discuss interpretations of probability notable for their diversity which is similar to the diversity of interpretations of a quantum state.

[1]Now this theory is practically forgotten. However, it played an important role in search for an adequate axiomatics of probability theory and randomness, especially von Mises' *principle of randomness*. We proceed with the Kolmogorov theory, see my monographs [133], [156] for von Mises probability versus quantum probability.

Already in Chapter 1 we derive a version of the famous *Bell inequality* [31] (in the Wigner form) as expressing two basic properties of a measure: additivity and non-negativity. The derivation is based on the assumption of the existence of *a single probability measure* serving to represent all probability distributions involved in this inequality. We remark that Kolmogorov endowed his model of probability with a "protocol" of its application: each complex of experimental conditions (i.e., each context) is described by its own probability space. Thus in any multi-contextual experiment, such as experiments on Bell's inequality, we are dealing, in general, with a family of probabilities corresponding to different contexts. Kolmogorov studied the problem of the existence of the common probability space for stochastic processes and found the corresponding necessary and sufficient conditions.

Chapter 3 may be difficult for physicists. Here we present the standard construction of the *Lebesgue extension* of a countably additive measure which is originally defined on a simple system of sets. An example of *non-measurable set* is of foundational interest. It contradicts (physical) intuition that probability can be assigned to any event.[2] Moreover, its existence is based the *axiom of choice* (E. Zermelo, 1904). The formulation of this axiom taken by itself sounds still acceptable. However, it has some equivalent formulations, e.g., one known as "the well-ordering theorem", which are really counter-intuitive. Some mathematicians are suspicious of this axiom. Our aim was to show that the foundations of classical (measure-theoretic) probability are more ambiguous than the foundations of quantum (complex Hilbert space) probability. The last section of Chapter 3 presents "exotic generalization of concept of probability" such as *negative probability* (cf. Dirac, Feynman, Aspect) and p-adic probability with possible applications in quantum foundations.

Chapter 4 contains the basics of the quantum formalism. This chapter plays the introductory role for a newcomer to quantum theory, but it can also be interesting for physicists. Here we proceed by using general theory of *quantum instruments*.

Chapter 5 gets to the core of classical versus quantum probability interplay. It starts with Feynman's analysis of the probability structure of the *two-slit experiment* [88]. His conclusion is that classical probability is not applicable to results of the multi-contextual structure of this experiment.

[2]This non-measurability argument was explored by I. Pitowsky in his analysis of violation of Bell's inequality [218]. However, nowadays it is completely forgotten. Nobody would say: "Bell's inequality is violated because some sets of hidden variables are not measurable."

The rest of the chapter is devoted to the mathematical formalization of this contextual probability viewpoint.[3]

In my previous publications, e.g., [156], I treated contextual probability as *non-Kolmogorovian probability*, by following Accardi (see [2] - [4] - unfortunately, he was not able to publish his book in English).[4] However, now I understand that this terminology has to be used with caution. Formally, probabilistic data from two-slit experiment and Bell's inequality experiment can be embedded in the classical probability space. However, this embedding is not straightforward: probabilities have to be treated as conditional, see Chapter 8 for construction of this embedding for Bell's experiment.

For me, Bell's argument sounds as follows: *we cannot represent probabilities collected for different pairs of orientations of polarization beam splitters as unconditional classical probabilities.* However, I am not sure that Bell would accept this interpretation. He was concentrated on the nonlocality dimension - by trying to justify Bohmian mechanics as the genuine quantum model.[5]

Chapter 6 is the most difficult for reading. It is about interpretations of quantum mechanics. The main problem is their diversity. My attempt to classify them may be found boring. One can just scan this chapter: the classical interpretations of von Neumann and Einstein-Ballentine and modern ones such as the information interpretation (Zeilinger-Brukner), statistical Copenhagen interpretation (Plotnitsky), QBism (Fuchs, Schack, Mermin), and the Växjö interpretation (Khrennikov). Zeilinger, Brukner and Plotnitsky can be considered as neo-Copenhagenists. QBists are also often treated in the same way. However, this is the wrong viewpoint on QBism. The Växjö interpretation can be considered as merging the Einstein-Ballentine ensemble interpretation and Bohr's contextual viewpoint on quantum

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[3] We remark that R. Feynman appealed to the two-slit experiment in all his discussions on quantum foundations. Similarly to N. Bohr, he considered this experiment as the heart of quantum mechanics (we remark that the same point was permanently expressed in publications and talks of L. Accardi). I share this viewpoint of Bohr-Feynman-Accardi. On the other hand, nowadays one can often hear that entanglement and violation of Bell's inequality (rather than interference demonstrated in the two-slit experiment) are the key elements of quantum theory. I do not think so as from the contextual viewpoint the two-slit and Bell experiments are of the same nature. Mathematically both are expressed as violations of theorems of Kolmogorovian probability theory: the *formula of total probability* (the two-slit experiment) and the Bell inequality.

[4] Feynman [88] did not hear about Kolmogorov's axiomatics of probability theory; he wrote about violation of laws of Laplacian probability theory.

[5] Chapter 8 may do harm for a young physicist's attitude towards the nonlocality problem. If the idea of quantum nonlocality is dear to the reader, probably just skip this chapter.

observables. Surprisingly, this interpretation has a lot in common with QBism (see Chapter 6).

Chapter 7 is devoted to quantum randomness. This is mainly a philosophic discussion about inter-relation of von Neumann's irreducible randomness and classical approaches to randomness. Finally, we discuss possible applications of quantum probability outside of physics, cognition, psychology, biology, economics, Chapter 9. This chapter is of introductory character and its aim is just to inform the reader about such applications.

I hope that the book will serve as a textbook on classical and quantum probability and randomness (Chapters 1, 2, 4, 5, 7) and interpretations of quantum mechanics (Chapter 6). Chapter 8 presents the author's viewpoint on Bell's inequality and Chapter 9 informs readers about new areas of application of the quantum formalism: biology, cognition, economics.

This book combines short mathematical introductions to probability and randomness with rather long discussions on their interpretations. Readers who are not so much interested in the latter can simply skip interpretational parts.

I would like to thank I. Basieva, C. Fuchs, A. Plotnitsky, and Zeilinger for numerous critical discussions on interpretational issues. Their views differ crucially from my own (and from each other) and such discussions were fruitful for me. This is a good occasion to thank once again A. Bulinsky and A. Shyryaev who explained to me that Kolmogorov's viewpoint on probability was *contextual*: each complex of experimental conditions determines its own probability space, see section 2 in [177].

Vienna – Växjö, 2013–2015.

Contents

Chapter 1

Foundations of Probability

We start with the remark that, in contrast with, e.g., geometry, axiomatic probability theory was created not so long ago. Soviet mathematician Andrei Nikolaevich Kolmogorov presented the modern axiomatics of probability theory only in 1933 in his book [177]. The book was originally published in German.[1] The English translation [178] was published only in 1952 (and the complete Russian translation [179] of the German version [177] only in 1974).[2] Absence of an English translation soon (when the German language lost its international dimension) led to the following problem. The majority of the probability theory community did not have a possibility to read Kolmogorov. Their picture of the Kolmogorov model was based on its representations in English (and Russian) language textbooks. Unfortunately, in such representations a few basic ideas of Kolmogorov disappeared, since they were considered as philosophical remarks with no direct relevance to mathematics. This is partially correct, but *probability theory is not just mathematics.* It is a physical theory and, as any physical theory, its mathematical formalism has to be endowed with some interpretation. In Kolmogorov's book the interpretation question was discussed in very detail. However, in the majority of mathematical representations of Kolmogorov's approach, the interpretation issue is not enlightened at all. From my viewpoint, one of the main negative consequences of this ignorance was oblivion of *contextuality* of Kolmogorov's theory of probability. Kolmogorov designed his probability theory as a mathematical formalization of random experiments (see also discussion below). For him, each exper-

[1] The language and the publisher (Springer) were chosen by a rather pragmatic reason. Springer paid in gold, young Kolmogorov felt the need of money, and gold was valuable even in the Soviet Union.

[2] The first Russian version was published in 1936 [176]. However, it was not identical to the original German version, in some places it was shorten.

imental context C generates its own probability space (endowed with its own probability measure). It is practically impossible to find a probability theory textbook mentioning this key interpretational issue of Kolmogorov's theory. We remark that contextuality of classical probability theory plays very important role when classical and quantum probability theories are being compared, see Chapter 5.

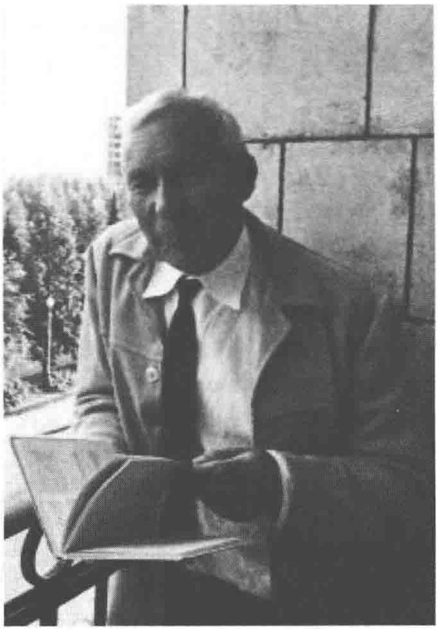

Fig. 1.1 Andrei Nikolaevich Kolmogorov

We recall that at the beginning of 20th century probability was considered as a part of mathematical physics and not pure mathematics. In 1900 at the Paris mathematical congress David Hilbert presented the famous list of problems [114] - [116]. The 6th problem is about axiomatization or physical theories:

"Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics."

It was not clear what features of nature have to be incorporated in