Advanced Mathematics

(2nd Edition) (Ⅰ)

北京邮电大学高等数学双语教学组　编

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普通高等教育“十三五”规划教材

Advanced Mathematics

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内 容 简 介

本书是根据国家教育部非数学专业数学基础课教学指导分委员会制定的工科类本科数学基础课程教学基本要求编写的全英文教材，全书分为上、下两册，此为上册，主要包括函数与极限，一元函数微积分及其应用和微分方程三部分。本书对基本概念的叙述清晰准确，对基本理论的论述简明易懂，例题习题的选配典型多样，强调基本运算能力的培养及理论的实际应用。本书可作为高等理工科院校非数学类专业本科生的教材，也可供其他专业选用和社会读者阅读。

The aim of this book is to meet the requirement of bilingual teaching of advanced mathematics. This book is divided into two volumes, and the first volume contains functions and limits, calculus of functions of a single variable and differential equations. The selection of the contents is in accordance with the fundamental requirements of teaching issued by the Ministry of Education of China and based on the property of our university. This book may be used as a textbook for undergraduate students in the science and engineering schools whose majors are not mathematics, and may also be suitable to the readers at the same level.

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前　　言

高等数学(微积分)是一门研究运动和变化的数学,产生于16世纪至17世纪,是受当时科学家们在研究力学问题时对相关数学的需要而逐渐发展起来的。在高等数学中,微分处理的是当函数已知时,如何求该函数变化率的问题,如曲线的斜率、运动物体的速度和加速度等;而积分处理的是当函数的变化率已知时,如何求该函数的问题,如通过物体当前的位置及作用在该物体上的力来预测该物体的未来位置,计算不规则平面区域的面积,计算曲线的长度等。现在,高等数学已经成为高等院校学生尤其是工科学生最重要的数学基础课程之一,学生在这门课程上学习情况的好坏对其后续课程能否顺利学习有着至关重要的影响。

本书第二版是在第一版的基础上,根据北邮高等数学双语教学组多年的教学实践及第一版教材的使用情况进行全面修订而成。本书上册各章节具体的撰写分工如下:第一章由艾文宝教授编写,第二章和第三章由李晓花副教授编写,第四章和第五章由袁健华教授编写,第六章由默会霞副教授编写。全书由艾文宝教授进行内容审核。本书在内容编排和讲解上适当吸收了欧美国家微积分教材的一些优点,新版教材尽量做到逻辑严谨、叙述清晰、直观性强、例题丰富。本套教材中文版、英文版及习题解答是相互配套的,特别适合双语高等数学的教学需要。由于作者水平有限,加上时间匆忙,书中出现一些错误在所难免,欢迎并感谢读者通过邮箱 jianhuayuan@bupt.edu.cn 指出错误,以便我们及时纠正。

编　者

Preface

Advanced mathematics that we refer to contains mainly calculus. Calculus is the mathematics of motion and change. It was first invented to meet the mathematical needs of the scientists of the sixteenth and seventeenth centuries, and the needs that were mainly mechanical in nature. Differential calculus deals with the problem of calculating rates of change. It enables people to define slopes of curves, to calculate velocities and accelerations of moving bodies etc.. Integral calculus deals with the problem of determining a function from information about its rate of change. It enables people to calculate the future location of a body from its present position and a knowledge of the forces acting on it, to find the areas of irregular regions in the plane, to measure the lengths of curves, and so on. Now advanced mathematics becomes one of the most important courses of the college students in natural science and engineering.

The second edition of the book is revised based on implementation experience of its first edition. The contents of the book are written by the authors as follows: Professor Wenbao Ai for the first chapter, Associate professor Xiaohua Li for the second and third chapter, Professor Jianhua Yuan for the fourth and fifth chapter, and associate professor Huixia Mo for the sixth chapter. The book of new edition is contributed as logically and intuitively as possible. Its Chinese and English versions and a corresponding exercise book form a family-united system, which is very useful to the bilingual-teaching. For any errors remaining in the book, the authors would be grateful if they were sent to: jianhuayuan@bupt.edu.cn.

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Chapter 1

Fundamental Knowledge of Calculus

Calculus is the mathematics of motion and change. Its object of study is about variables. Dependent relations between variables are described by functions. And limits are a basic tool for study of functions. In this chapter, we shall introduce fundamental knowledge of calculus, such as sets, functions, limits, continuity, and so on.

1.1 Mappings and Functions

1.1.1 Sets and Their Operations

1. The concept of set

In mathematics, a **set** [集合, 简称集] is a well-defined collection of distinct objects. The objects that make up a set, also known as the **elements** [元素] of a set, can be anything: numbers, people, letters of the alphabet, other sets, and so on. Specifically, a set is a gathering together into a whole of definite, distinct objects—which are called elements of the set.

Sets are conventionally denoted with capital letters, such as A, B, C, and so on. And elements are denoted with lowercase letters, such as a, b, c, and so on. If A is a set and a is one of the objects of A, this is denoted by $a \in A$, and is read as "a **belongs** to A", or "a is an element of A". If a is not an element of A then this is written as $a \notin A$ or $a \overline{\in} A$, and is read as "a **does not belong to** A".

There are two ways of describing a set. One way is by enumeration, and the other way is by intension. For example, A is the set of four seasons of a year, which can be written as

$$A = \{\text{spring}, \text{summer}, \text{autumn}, \text{winter}\}.$$

Let B denote the set of all the positive integers between 1 and 100, which can be written as

$$B = \{1, 2, 3, 4, \cdots, 98, 99, 100\}.$$

In the above two examples, the sets are specified by listing each element of the set—called enumeration. However, not all sets can be represented by enumeration. In this case, the set can be specified by intension, i. e., using a rule or semantic description to indicate the properties of elements. If S is the set of all the elements satisfying property P, then it can be written as

$$S=\{x \mid x \text{ has property } P\}.$$

For example, C is the set of all carnivores, which can be written as

$$C=\{x \mid x \text{ is a carnivore}\}.$$

Let D denote the set of all solutions to the equation $x^2-4=0$, which can be written as

$$D=\{x \mid x^2-4=0\}.$$

If a set S is composed of n elements, then S is a **finite set** [有限集], otherwise, it is an **infinite set** [无限集], where n is a certain natural number. For example, both sets $A=\{a,b,c,d\}$ and $B=\{1,2,3,4,5\}$ are finite sets, and the set $C=\{x \mid x \text{ is a real number greater than } 1\}$ is an infinite set.

Here we show some commonly used sets of numbers as follows:

$\mathbf{N}=\{x \mid x \text{ is a natural number}\}=\{0,1,2,\cdots\}$;

$\mathbf{N}_+=\{x \mid x \text{ is a positive natural number}\}=\{1,2,\cdots\}$;

$\mathbf{Z}=\{x \mid x \text{ is an integer}\}=\{0,\pm 1, \pm 2,\cdots\}$;

$\mathbf{Q}=\{x \mid x \text{ is a rational number}\}$;

$\mathbf{R}=\{x \mid x \text{ is a real number}\}$.

Note that sometimes we mark "$*$" at the top-right corner of the letter symbol of a number set to denote the number set without the element 0, and mark "$+$" in the subscript to denote the number set without the negative numbers. For example, $\mathbf{R}^*$ denotes the set of all real numbers without 0, and $\mathbf{R}_+$ denotes the set of all nonnegative real numbers.

Given sets A and B, A is called a **subset** [子集] of B, if every element of A is also an element of B. It is notated by $A\subseteq B$ (read as "A is contained in B", see Figure 1.1.1(a)). Equivalently, we can write $B\supseteq A$, read as "B includes A", or "B contains A". If A is not a subset of B, then we write $A\nsubseteq B$ (Figure 1.1.1(b)). For example, we have $\mathbf{N}\subseteq\mathbf{Q}\subseteq\mathbf{R}$ and $\mathbf{Q}\nsubseteq\mathbf{R}_+$.

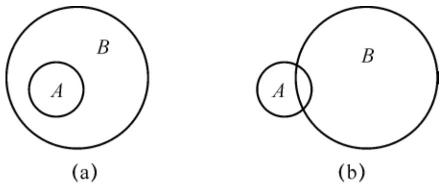

Figure 1.1.1

If both sets A and B are subsets of each other, i.e., $A\subseteq B$ and $B\subseteq A$, then the sets A and B are **equal** [相等], notated as $A=B$. If A is not equal to B, then we write $A\neq B$. For example, suppose that

$$A=\{-1,1\}, \quad B=\{x \mid x^2-1=0\},$$

then we have $A=B$.

If two sets A and B satisfy $A\subseteq B$ and $A\neq B$, then A is called a **proper subset** [真子集] of B, written as $A\subset B$. For example, $\mathbf{N}\subset\mathbf{Z}$, $\mathbf{Z}\subset\mathbf{Q}$ and $\mathbf{Q}\subset\mathbf{R}$.

An **empty set** or **null set** [空集] is a set with no elements. It can be symbolized with $\varnothing$.

For example, the real solution set for the equation $x^2+1=0$,

$$\{x \mid x\in \mathbf{R} \text{ and } x^2+1=0\},$$

is an empty set. It is agreed that the empty set is a subset of any set, i. e. , for any set A, there is $\varnothing\subseteq A$.

Example 1. 1. 1 Find all the subsets of the set $A=\{1,2,3\}$.

Solution $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{2,3\}$, $\{1,3\}$, $\{1,2,3\}$. ■

Example 1. 1. 2 Let $A=\{-2,-1,1,2\}$ and $B=\{x \mid x^3-x^2-4x+4=0, x\in \mathbf{R}\}$. Determine if the relationship $A=B$ is ture.

Solution As the real solutions to the equation $x^3-x^2-4x+4=0$ are $x_1=1$, $x_2=2$, and $x_3=-2$, we have

$$B=\{1,2,-2\}.$$

Then there are $B\subseteq A$ and $A\nsubseteq B$. Therefore, $A\neq B$. ■

2. Set operations

There are four fundamental operations of sets: **union** [并], **intersection** [交], **difference** [差] and **complement** [补].

(1) Union: $A\cup B$

Given sets A and B, the **union** [并集] of A and B, denoted by $A\cup B$ (see Figure 1. 1. 2 (a)), is the set of all elements that are in A or in B.

$$A\cup B=\{x \mid x\in A \text{ or } x\in B\}.$$

For example,

$$\{1,2,3\}\cup\{2,3,4\}=\{1,2,3,4\};$$

$$\{x \mid x\in \mathbf{R} \text{ and } x\leqslant 0\}\cup\{x \mid x\in \mathbf{R} \text{ and } x\geqslant 0\}=\mathbf{R}.$$

(2) Intersection: $A\cap B$

Given sets A and B, the **intersection** [交集] of A and B, denoted by $A\cap B$ (see Figure 1. 1. 2(b)), is the set of all elements that are both in A and in B.

$$A\cap B=\{x \mid x\in A \text{ and } x\in B\}.$$

For example,

$$\{1,2,3\}\cap\{2,3,4\}=\{2,3\};$$

$$\{x \mid x\in \mathbf{R} \text{ and } x\leqslant 0\}\cap\{x \mid x\in \mathbf{R} \text{ and } x\geqslant 0\}=\{0\}.$$

(3) Difference: $A-B$

Given sets A and B, the **difference** [差集] of A and B, denoted by $A-B$ or $A\backslash B$ (see Figure 1. 1. 2(c)), is the set of all elements that are members of A but not members of B.

$$A-B=\{x \mid x\in A \text{ and } x\notin B\}.$$

For example,

$$\{1,2,3\}-\{2,3,4\}=\{1\};$$

$$\{x \mid x\in \mathbf{R} \text{ and } x\leqslant 2\}-\{x \mid x\in \mathbf{R} \text{ and } x>0\}=\{x \mid x\in \mathbf{R} \text{ and } x\leqslant 0\}.$$

(4) Complement: A^C

In certain settings, all sets under discussion are considered to be subsets of a given **universal set** [全集] X. In such cases, the **complement** [补集,余集] of a set A, denoted by A^C (Figure 1. 1. 2 (d)), is defined as

$$A^C = X - A.$$

For example, the complement of the set $A=\{x \mid 0<x\leqslant 1\}$ is

$$A^C=\{x \mid x\leqslant 0 \text{ or } x>1\}.$$

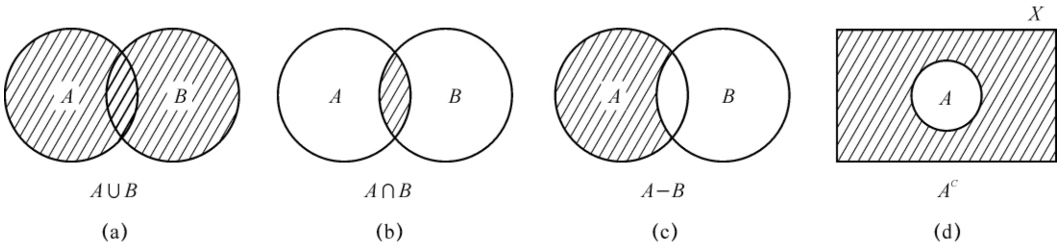

Figure 1. 1. 2

(5) Fundamental rules of set operations

Theorem 1. 1. 1 (Rules of set operations) Let A, B and C be three sets. There are

① **Commutative law** [交换律] $A\cup B=B\cup A$; $A\cap B=B\cap A$.

② **Associative law** [结合律] $(A\cup B)\cup C=A\cup(B\cup C)$; $(A\cap B)\cap C=A\cap(B\cap C)$.

③ **Distributive law** [分配律] $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$;

$(A\cap B)\cup C=(A\cup C)\cap(B\cup C)$;

$(A\backslash B)\cap C=(A\cap C)\backslash(B\cap C)$.

④ **Idempotent law** [幂等律] $A\cup A=A$; $A\cap A=A$.

⑤ **Absorption law** [吸收律] $A\cup\varnothing=A$, $A\cap\varnothing=\varnothing$. If $A\subseteq B$, then $A\cup B=B$ and $A\cap B=A$.

The above rules can be verified by the definition of equality of sets. Here we present the proof of $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ in the distributive law, and the rest is left to the readers.

Example 1. 1. 3 Let A, B and C be three sets. Prove $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

Proof We first try to prove $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.

$x\in A\cap(B\cup C)\Rightarrow x\in A$ and $x\in B\cup C$,

$\Rightarrow x\in A$ and "$x\in B$ or $x\in C$",

$\Rightarrow$"$x\in A$ and $x\in B$" or "$x\in A$ and $x\in C$",

$\Rightarrow x\in A\cap B$ or $x\in A\cap C$,

$\Rightarrow x\in(A\cap B)\cup(A\cap C)$.

Then we shall prove $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$.

$x\in(A\cap B)\cup(A\cap C)\Rightarrow x\in A\cap B$ or $x\in A\cap C$,

$\Rightarrow$"$x\in A$ and $x\in B$" or "$x\in A$ and $x\in C$",

$\Rightarrow x\in A$ and "$x\in B$ or $x\in C$",

$\Rightarrow x\in A$ and $x\in B\cup C$,

$\Rightarrow x\in A\cap(B\cup C)$.

Hence, $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$. ■

Note that the symbol "$\Rightarrow$" represents "deduce" (or "imply") in the above proof. If we replace the symbol "$\Rightarrow$" with the symbol "$\Leftrightarrow$" (be equivalent to) in the proof of "$A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$", then the second part of the proof can be accomplished.

(6) The Cartesian product of sets

In set theory, a **Cartesian product** [笛卡儿积] is a mathematical operation that returns a set from multiple sets. That is, for two sets A and B, the Cartesian product, denoted by $A\times B$, is the set of all ordered pairs (x, y), where $x\in A$ and $y\in B$, i. e.,

$$A\times B=\{(x,y)\mid x\in A \text{ and } x\in B\}.$$

For example, $\mathbf{R}\times\mathbf{R}=\{(x,y)\mid x\in\mathbf{R},\ y\in\mathbf{R}\}$ is the set of all points in the xOy-plane. $\mathbf{R}\times\mathbf{R}$ is usually written as $\mathbf{R}^2$. And an n-dimensional space is denoted by

$$\mathbf{R}^n=\{(x_1,x_2,\cdots,x_n)\mid x_1,x_2,\cdots,x_n\in\mathbf{R}\}.$$

3. Intervals and neighborhoods

We focus on **sets of real numbers**, i. e., subsets of **R**. An interval is a set of real numbers between two other numbers, and is a widely used class of sets of real numbers. Let a and b be two real numbers, and $a\leqslant b$. The closed interval $[a,b]$, the open interval (a,b), and the half-open intervals $[a,b)$ and $(a,b]$ are the following sets of real numbers:

$$[a,b]=\{x\mid a\leqslant x\leqslant b\};$$

$$(a,b)=\{x\mid a<x<b\};$$

$$[a,b)=\{x\mid a\leqslant x<b\};$$

$$(a,b]=\{x\mid a<x\leqslant b\}.$$

Here, a and b are the endpoints of intervals. For the open interval (a,b), there are $a\notin(a,b)$ and $b\notin(a,b)$. The above four intervals are bounded intervals and $b-a$ is called the length of the intervals. Bounded intervals are the segments with finite lengths on the number axis (see Figure 1.1.3).

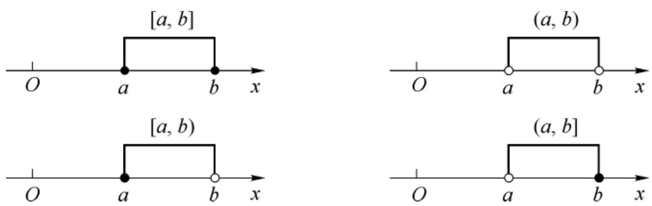

Figure 1.1.3

Besides, there are intervals whose endpoints are $\pm\infty$, and we call these intervals unbounded intervals. For example,

$$[a,+\infty)=\{x\mid x\geqslant a\};$$

$$(-\infty,b)=\{x\mid x<b\};$$

$$(-\infty,+\infty)=\{x\mid x\in\mathbf{R}\}=\mathbf{R}.$$

Here "$+\infty$" and "$-\infty$" are read as "positive infinity" and "negative infinity", respectively. The unbounded intervals $[a,+\infty)$ and $(-\infty,b)$ are shown in Figure 1.1.4.

Figure 1.1.4

In later discussions, if we do not focus on whether the interval contains its endpoints, and whether it is a bounded interval or not, then we simply call it an "**interval**" [区间], denoted by the symbol I.

Another widely used concept is **neighborhood** [邻域]. Any open interval with the center (midpoint) a is called the neighborhood of a, and we write it as $U(a)$. Let $a\in \mathbf{R}$, $\delta>0$ and the open interval $(a-\delta, a+\delta)$ is a neighborhood of a, which is called the δ neighborhood of a. It is the set of all real numbers whose distance from a is less than δ, denoted by $U(a,\delta)$, that is

$$U(a,\delta)=\{x \mid a-\delta<x<a+\delta\}=\{x \mid |x-a|<\delta\}.$$

Here a is the center of the neighborhood, and δ is its radius (see Figure 1.1.5).

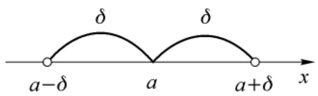

Figure 1.1.5

The set $U(a,\delta)-\{a\}$ is the δ neighborhood of a without the number a, which is called the **deleted $\boldsymbol{\delta}$ neighborhood of** a and denoted by $\mathring{U}(a,\delta)$, that is

$$\mathring{U}(a,\delta)=\{x \mid 0<|x-a|<\delta\}.$$

For convenience, we sometimes say that the open interval $(a-\delta, a)$ is the **$\boldsymbol{\delta}$ left neighborhood** of a and the open interval $(a, a+\delta)$ is the **$\boldsymbol{\delta}$ right neighborhood** of a.

1.1.2 Mappings and Functions

***1. The concept of mapping**

Definition 1.1.1 (Mapping [映射]) Let A and B be two nonempty sets. If there exists such a rule f that every element x in A is associated with one unique element y in B under the rule f, then f is called a **mapping** [映射] from A to B, which is denoted by

$$f: A\to B,$$

or

$$f: x\to y=f(x), \quad x\in A.$$

Here y is called the **image** [像] of x under the mapping f, and x is called the **inverse image** [原像] of y under the mapping f. The set A is called the **domain** (or **domain of definition**) [定义域] of the mapping f, denoted by D_f, i.e., $D_f=A$. The set composed of the image points of all the elements in A is called the **range** [值域] of the mapping f, denoted by R_f or $f(A)$, i.e.,

$$R_f=f(A)=\{f(x) \mid x\in A\}.$$

Note (1) Three essential factors of a mapping are the domain A, the superset B of the range, and the rule f.