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FIXED POINT THEOREMS

BY

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Preface

This book is intended as an introduction to fixed point theorems and to their applications in analysis. Apart from applicable theorems, I have included those which interested me.

Since applications usually involve spaces of functions, I give Banach space versions of most of the theorems. The book is thus aimed at readers with a general interest in functional analysis. However, I have hardly touched on a series of recent developments, by F. E. Browder and others; for these see Browder's forthcoming book. To fill obvious gaps in the direction of pure topology, the reader can refer to the excellent surveys by van der Walt (1963), Bing (1969), and Fadell (1970), and to Brown's book (1970).

The methods of proof are those which will seem natural to the functional analyst. Most of the results are derived from Brouwer's fixed point theorem for the ball B^n; to get this theorem we use some facts about homology groups. After that, algebraic topology is rarely mentioned. I have chosen to give geometric proofs, rather than that to develop degree theory and base everything on that. The degree and other invariants are discussed at the end of the book.

I must thank Dr F. Smithies for suggesting that I write this book, and for helpful comments on successive versions of it. I am also indebted to the University of Cape Town for two periods of study leave during which much of the writing was done, and to the Universities of Cambridge, Edinburgh and Wales (Swansea) for their hospitality on these occasions. I should mention, too, the contribution of various members of those universities who attended my lectures at a time when I was still sorting out my ideas.

However, my greatest helper has been Daphne, who provided conditions in which I felt like working, and who spent many hours typing the manuscript.

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Cape Town
March 1973

Symbols used

$\mathscr{M}^0$	Interior of a set $\mathscr{M}$
$\overline{\mathscr{M}}$	Closure of $\mathscr{M}$
$\partial\mathscr{M}$	Boundary of $\mathscr{M}$
$\mathrm{co}(\mathscr{M})$	Convex cover of $\mathscr{M}$
$\overline{\mathrm{co}}\,(\mathscr{M})$	Closed convex cover of $\mathscr{M}$
$C\,(\mathscr{M})$	Space of continuous bounded functions on $\mathscr{M}$
$\chi(\mathscr{M})$	Measure of noncompactness (p. 32)
$\mathrm{span}\,(\mathscr{M})$	Linear subspace spanned by $\mathscr{M}$
l^2	Space of square-summable sequences
$\mathscr{H}_0$	Hilbert cube (p. 13)
$\mathscr{D}(T)$	Domain of a mapping T
$\mathscr{R}(T)$	Range of T
$\mathscr{G}(T)$	Graph of T
$F(T)$	Set of fixed points of T
$N(x,\epsilon)$	ϵ-neighbourhood of x
$(\cdot,\cdot)$	Inner product
$\varnothing$	Empty set
I	Identity operator
$\mathscr{B}^*$	Dual of Banach space $\mathscr{B}$
Z	Group of integers
deg ( )	Degree (pp. 77, 80)
rot ( )	Rotation (p. 75)

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1. *Contraction mappings*

1.1 Introduction

Consider a mapping T of a set $\mathscr{M}$ into $\mathscr{M}$ (or into some set containing $\mathscr{M}$). One of the few questions we can ask (in this general setting) is whether some point is mapped onto itself; that is, does the equation

$$Tx = x$$

have a solution? If so, x is called a *fixed point* of T. The theorems we prove assert that, under suitable conditions on $\mathscr{M}$ and T, a fixed point exists.

Obviously, the conditions must always imply that $\mathscr{M} \neq \varnothing$. Usually, $\mathscr{M}$ is a topological space and some conditions of continuity and compactness (or at least completeness) are needed.

We shall see that many existence theorems of analysis can be treated as special cases of suitable fixed point theorems.

In the present chapter, we place rather strong conditions on T and rather weak conditions on $\mathscr{M}$. Because of the simplicity of its assumptions, 1.2.2 is the most widely applied fixed point theorem. We discuss some of its applications in §§1.3, 1.4, 6.2 and 6.5.

We first give a few simple and general results.

THEOREM 1.1.1 *If T maps $\mathscr{M}$ into $\mathscr{M}$ then any fixed point z of T is in $\cap T^n\mathscr{M}$. Conversely, if $\cap T^n\mathscr{M} = \{y\}$, a one-point set, then y is a fixed point for T.*

Proof. Since Ty must be in $\cap T^n\mathscr{M}$, we have $Ty = y$. □

THEOREM 1.1.2 (*Principle of successive approximations*) *If T is continuous on a Hausdorff topological space $\mathscr{M}$ to $\mathscr{M}$ and if $\lim T^n x = y$ exists then $Ty = y$.*

Proof. $Ty = T(\lim T^n x) = \lim T^{n+1}x = y$. □

THEOREM 1.1.3 *Let $\mathscr{U}$ be a metric space. Suppose that T is a continuous mapping of (a closed subset of) $\mathscr{U}$ into a compact subset of $\mathscr{U}$ and that, for each $\epsilon > 0$, there exists $x(\epsilon)$ such that*

$$\rho(Tx(\epsilon), \quad x(\epsilon)) < \epsilon. \tag{1}$$

Then T has a fixed point.

Proof. Let T map the closed subset $\mathscr{M}$ into the compact subset $\mathscr{Z}$. Since $Tx(\epsilon)$ is in $\mathscr{Z}$ we can assume that for some sequence $\epsilon_n \to 0$ we have $Tx(\epsilon_n) \to y \in \mathscr{Z}$. By (1) we also have $x(\epsilon_n) \to y$ so that $y \in \mathscr{M}$. Thus Ty is defined and

$$Ty = T(\lim x(\epsilon_n)) = \lim Tx(\epsilon_n) = y. \quad \square$$

DEFINITION 1.1.4 The points $x(\epsilon)$ satisfying (1) will be called *ϵ-fixed points* for T.

We shall often use 1.1.3 to obtain fixed points from ϵ-fixed points. However, the usual position is that we can obtain the ϵ-fixed points by a constructive argument. Theorem 1.1.3 does not give a constructive proof for the existence of fixed points. Brouwer (1952) argues that only ϵ-fixed points have meaning for the intuitionist.

1.2 The contraction mapping theorem

DEFINITION 1.2.1 Let T be a mapping of a metric space $\mathscr{M}$ into $\mathscr{M}$. We say that T is a *contraction mapping* if there exists a number k such that $0 < k < 1$ and

$$\rho(Tx, Ty) \leqslant k\rho(x, y) \quad (\forall x, y \in \mathscr{M}). \tag{1}$$

The following result is called the *Contraction Mapping Theorem.*

THEOREM 1.2.2 (Banach, 1922) *Any contraction mapping of a complete non-empty metric space $\mathscr{M}$ into $\mathscr{M}$ has a unique fixed point in $\mathscr{M}$.*

Proof. Let the mapping T satisfy (1) for some $k < 1$. Choose any point y in $\mathscr{M}$. The sequence of points $T^n y$ satisfies, for $n > 0$,

$$\rho(T^n y, T^{n+1} y) \leqslant k\rho(T^{n-1} y, T^n y),$$

so that by induction

$$\rho(T^n y, T^{n+1} y) \leqslant k^n \rho(y, Ty).$$

By the triangle inequality we have for $m \geqslant n$

$$\begin{aligned}\rho(T^n y, T^m y) &\leqslant \rho(T^n y, T^{n+1}y)+\rho(T^{n+1}, T^{n+2}y)+\ldots+\rho(T^{m-1}y, T^m y)\\ &\leqslant (k^n+k^{n+1}+\ldots+k^{m-1})\,\rho(y, Ty)\\ &\leqslant k^n(1-k)^{-1}\rho(y, Ty). \qquad \text{(A)}\end{aligned}$$

Thus $\rho(T^n y, T^m y)\to 0$ if $m, n\to\infty$. Since $\mathscr{M}$ is complete the sequence $T^n y$ has a limit z in $\mathscr{M}$. By 1.1.2, z is a fixed point for T. This fixed point is unique since if $Tz = z$ and $Tw = w$ we have

$$\rho(z, w) = \rho(Tz, Tw) \leqslant k\rho(z, w)$$

so that $\rho(z, w) = 0$; that is, $z = w$. □

For applications of 1.2.2, some further facts are important.

REMARK 1.2.3 *Under the conditions of* 1.2.2:

(i) *the fixed point z can be calculated as* $\lim T^n y$ *for any y in* $\mathscr{M}$;

(ii) $\rho(T^n y, z) \leqslant k^n(1-k)^{-1}\rho(y, Ty)$;

(iii) *For any y in* $\mathscr{M}$, $\rho(y, z) \leqslant (1-k)^{-1}\rho(Ty, y)$.

Proof.

(i) is clear from the proof of 1.2.2;

(ii) follows by letting $m\to\infty$ in the inequality (A);

(iii) follows from (ii) or from the inequality

$$\rho(y, z) \leqslant \rho(y, Ty)+\rho(Ty, Tz) \leqslant \rho(y, Ty)+k\rho(y, z). \ \square$$

There is an alternative form of 1.2.2 in which the contraction mapping is only defined on a suitable neighbourhood of the point y which is taken as the first approximation. This is suggested by 1.2.3 (iii), which gives a neighbourhood of y in which the fixed point must lie. For details and applications of this alternative theorem, see Copson (1968).

1.3 The Cauchy–Lipschitz theorem

We use the contraction mapping theorem to establish an existence–uniqueness theorem for ordinary non-linear differential equations.

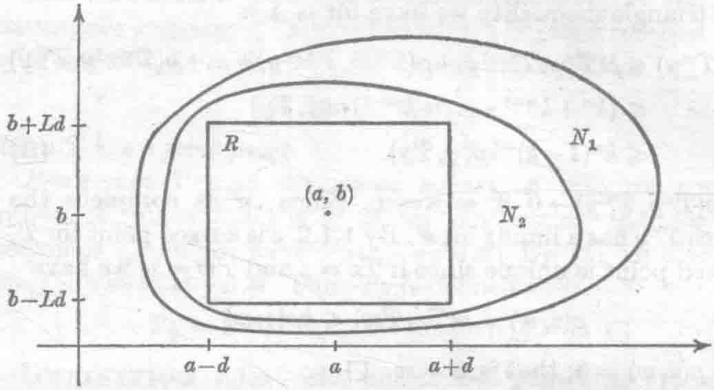

Fig. 1

THEOREM 1.3.1 (Lipschitz, 1876) *Let f be continuous and satisfy a Lipschitz condition with respect to y:*

$$|f(t,y)-f(t,z)| \leqslant K|y-z|$$

in some neighbourhood N_1 of (a,b). Then the differential equation with initial condition

$$\frac{dy}{dt} = f(t,y), \quad f(a) = b \tag{1}$$

has a unique solution in some neighbourhood of a.

Proof. We observe that (1) is equivalent to the integral equation

$$y(t) = b + \int_a^t f(x, y(x))\, dx \tag{2}$$

(for an approach which makes this transformation of the problem seem less accidental, see Chapter 5). We consider a set $\mathscr{M}$ of functions, and a mapping U in $\mathscr{M}$. The image Uy of a function y with values $y(x)$ will be given by

$$(Uy)(t) = b + \int_a^t f(x, y(x))\, dx. \tag{3}$$

How can we find a set of functions which is mapped into itself by U? We first choose a compact neighbourhood N_2 of (a,b), inside N_1; then f is bounded on N_2, say

$$|f(x,y)| \leqslant L \quad ((x,y) \in N_2).$$

If y is a function with graph in N_2 we have

$$|Uy(t)-b| = \left|\int_a^t f(t,y(t))\,dt\right| \leqslant L|t-a|.$$

This means that if y is a continuous function defined for

$$|t-a| \leqslant d$$

for which $|y(t)-b| \leqslant Ld$, then Uy satisfies the same conditions. We must choose d small enough for the rectangle (figure 1)

$$R = \overline{N}(a,d) \times \overline{N}(b,Ld)$$

to be in N_2. We then define $\mathscr{M}$ to be the set of continuous functions with graphs in R, and our argument shows that $\mathscr{M}$ is mapped into itself by U. We use the upper bound norm on $\mathscr{M}$.

To ensure that U is a contraction mapping we should also arrange, in choosing d, that $dK < 1$. Then we have, for y and z in $\mathscr{M}$

$$\begin{aligned}|Uy(t)-Uz(t)| &= \left|\int_a^t f(x,y(x))-f(x,z(x))\,dx\right| \\ &\leqslant d\sup|f(x,y(x))-f(x,z(x))| \\ &\leqslant dK\sup|y(x)-z(x)|.\end{aligned}$$

Thus
$$\begin{aligned}\|Uy-Uz\| &= \sup_t|Uy(t)-Uz(t)| \\ &\leqslant dK\sup|y(x)-z(x)| = dK\,\|y-z\|,\end{aligned}$$

and since $dK < 1$, U is a contraction mapping. Thus by 1.2.2, U has a unique fixed point in $\mathscr{M}$. This means that there is a unique function in $\mathscr{M}$ which is a solution of (1). Since any solution of (1) is in $\mathscr{M}$ (for d sufficiently small), there is a unique solution of (1). □

1.4 Implicit functions

We give a second application of the contraction mapping theorem.

THEOREM 1.4.1 (*Implicit Function Theorem*) *Let N be a neighbourhood of a point (a,b) in R^2. Suppose that f is a continuous* *function of x and y in N and that $\partial f/\partial y$ exists in N and is continuous at (a, b). Then if*

(i) $\dfrac{\partial f}{\partial y}(a, b) \neq 0,$

(ii) $f(a, b) = 0,$

there is a unique continuous function y_0 on some neighbourhood of a, such that $f(x, y_0(x)) = 0$.

Proof. We write D_f for $\partial f(a, b)/\partial y$. We will look for a fixed point of a mapping defined by

$$Tz(x) = z(x) - D_f^{-1} f(x, z(x)).$$

(This mapping is suggested by the idea of finding $y_0(x)$ by Newton's method.) It is clear that if y is fixed we must have $f(x, y(x)) \equiv 0$. We will find a set of functions $\mathscr{M}$ such that T maps $\mathscr{M}$ into $\mathscr{M}$ and that T is a contraction mapping in $\mathscr{M}$. Within N we choose a closed rectangle

$$R = \overline{N}(a, \epsilon) \times \overline{N}(b, \delta)$$

small enough to give

$$\left| D_f^{-1} \frac{\partial f}{\partial y}(x, y) - 1 \right| < \tfrac{1}{2} \quad ((x, y) \in R),$$

$$|D_f^{-1} f(x, b)| < \tfrac{1}{2}\delta \quad (|x| \leqslant \epsilon).$$

Now write $C = C(\overline{N}(a, \epsilon))$ and put

$$\mathscr{M} = \{y \in C : y(a) = b, \ \ \|y - \beta\| \leqslant \delta\}$$

(where β is the function identically equal to b). Clearly T maps $\mathscr{M}$ into C. We have

$$\|T\beta - \beta\| = \|D_f^{-1} f(x, b)\| < \tfrac{1}{2}\delta.$$

For (x, y) in R we have

$$\left| \frac{\partial}{\partial y}(y - D_f^{-1} f(x, y)) \right| = \left| \left(1 - D_f^{-1} \frac{\partial}{\partial y} f(x, y) \right) \right| < \tfrac{1}{2}.$$

Thus by the lemma below, if y and z are in $\mathscr{M}$,

$$|Ty(x) - Tz(x)| \leqslant \tfrac{1}{2}|y(x) - z(x)| \quad (x \in \overline{N}(a, \epsilon)),$$

so that $\|Ty - Tz\| \leqslant \frac{1}{2}\|y-z\|$. Thus T is a contraction mapping. Also

$$\begin{aligned}\|Ty-\beta\| &\leqslant \|Ty-T\beta\| + \|T\beta-\beta\| \\ &\leqslant \tfrac{1}{2}\|y-\beta\| + \|T\beta-\beta\| \\ &< \tfrac{1}{2}\delta + \tfrac{1}{2}\delta = \delta\end{aligned}$$

so that T maps $\mathscr{M}$ into $\mathscr{M}$. Since $\mathscr{M}$ is complete, T has a unique fixed point in $\mathscr{M}$. Thus our problem has a unique solution which can be calculated by successive approximations, using the operator T and starting from any member of $\mathscr{M}$. □

LEMMA *If* $|\partial f/\partial y| \leqslant \frac{1}{2}$ *at all points between* (x, y) *and* (x, z) *then* $|f(x,y) - f(x,z)| \leqslant \frac{1}{2}|y-z|$.

Proof. Use the mean value theorem. □

The argument given above can be used to prove a far more general form of the implicit function theorem. If f maps $\mathscr{B}\times\mathscr{C}$ into $\mathscr{C}$, where $\mathscr{B}$ and $\mathscr{C}$ are Banach spaces we must interpret D_f as the Fréchet derivative of f at (a, b); we replace (i) by the condition that $(D_f)^{-1}$ exists. We interpret $\mathscr{M}$ as a space of continuous functions defined on a neighbourhood in $\mathscr{B}$ with values in $\mathscr{C}$. Since the lemma remains true for the Fréchet derivative, all details of the proof carry over directly to the general case. In particular, if f maps $R^m \times R^n$ into R^n we interpret D_f as an $n \times n$ matrix of partial derivatives with respect to the y-variables and replace (i) by the condition that this matrix has an inverse.

In the above argument each value of $y_0(x)$ is calculated by a variant of Newton's method. It is easy to adapt the argument to show that for each x in a neighbourhood of a, the value $y_0(x)$ (such that $f(x, y_0(x)) = 0$) can be found by the ordinary Newton's method. Thus the conditions of the theorem give sufficient conditions for Newton's method.

1.5 Other applications of Banach's theorem

Various applications of the contraction mapping theorem are given in Kolmogorov and Fomin (1957). These provide excellent illustrations of the use of fixed point theorems in analysis.

It is sometimes doubtful whether to refer to the contraction mapping theorem in a proof, or to carry out the discussion in terms of successive approximations. In linear problems we could just as well use the Neumann series (for example, in the applications in Kolmogorov and Fomin; or in Harris, Sibuya and Weinberg (1969)). On the other hand, in the discussion of the Schwarz alternating method in Courant and Hilbert (1962, p. 293), it is clear that the discussion could have been phrased in terms of a contraction mapping but in fact the convergence of the successive approximations is discussed directly. The only disadvantage – virtual repetition of the proof of the contraction mapping theorem – is slight (since this proof is short) and is compensated by the advantage of having an argument complete in itself.

In the case of the more difficult fixed point theorems which we will give later, there is a definite gain when the theorem is applied, since by appealing to a general theorem which depends on a deep argument, we can hope to avoid going through an argument of similar depth in the particular case to which the theorem is applied.

Exercises

1. Show (by 1.2.2) that there is a unique continuous function f on $[-1, 1]$ such that

$$f(x) = x + \tfrac{1}{2}\sin f(x).$$

(Consider continuous functions such that $|f(x)| \leqslant 2$.)

2. Extend 1.2.2 to the case where T^k is a contraction mapping for some integer $k > 1$ (see 5.2.1).

3. If $\mathscr{M}$ is a compact non-empty metric space, T maps $\mathscr{M}$ into $\mathscr{M}$ and $\rho(Tx, Ty) < \rho(x, y)$ for $x \neq y$, show that T has a unique fixed point (see 5.2.3).

4. If T is a contraction mapping of a Banach space $\mathscr{V}$ into itself, show that the equation $Tf - f = g$ has a unique solution f for each g in $\mathscr{V}$. Also show that $T - I$ and $(T-I)^{-1}$ are uniformly continuous (see 4.4.2).

5. If T is continuous in a metric space and $\rho(T^n x, T^{n+1}x) \to 0$ then any limit point of $\{T^n x\}$ is a fixed point of T.

2. *Fixed points in compact convex sets*

The principal results of this chapter are Brouwer's theorem (2.1.11), Schauder's theorem (2.3.7) and Tychonoff's theorem (2.3.8), all of which assert that every continuous mapping of a compact convex set into itself must have a fixed point. We end with an example showing that it is not enough for the set to be bounded, complete and convex.

2.1 The fixed point property

DEFINITION 2.1.1 A topological space $\mathscr{X}$ is said to possess the *fixed point property* if every continuous mapping of $\mathscr{X}$ into $\mathscr{X}$ has a fixed point.

It is often possible to decide that a set has *not* got the fixed point property, by finding a mapping without fixed points. (Consider, for instance, the real line or the unit circle.)

An elementary argument shows that the unit interval [0, 1] has the fixed point property. A fairly simple argument (see § 10.1) shows that the closed unit disc in the plane has the fixed point property. In all other important cases the fixed point property is rather hard to establish.

We observe first that the fixed point property is a topological property.

THEOREM 2.1.2 *If $\mathscr{X}$ is homeomorphic to $\mathscr{Y}$ and $\mathscr{X}$ has the fixed point property then $\mathscr{Y}$ has the fixed point property.*

Proof. An exercise. □

Using 2.1.2 and the results for the disc, one can show that various plane sets in amoeboid shapes have the fixed point property. But to deal with a spider's shape with a two-dimensional body and one-dimensional legs, or with a string of beads, one needs the next theorem.

DEFINITION 2.1.3 We say that $\mathscr{X}$ *is a retract of* $\mathscr{Y}$ if $\mathscr{X} \subset \mathscr{Y}$ and there exists a continuous mapping r of $\mathscr{Y}$ into $\mathscr{X}$ such that $r = I$ on $\mathscr{X}$. (We then call r a *retraction* mapping.)

EXAMPLE 2.1.4 *A closed convex non-empty subset* $\mathscr{X}$ *of* E^n *or of a Hilbert space is a retract of any larger subset.*

Sketch of proof. The required retraction mapping is obtained by mapping each point onto the nearest point of $\mathscr{X}$. For details see Bourbaki (1955, 5.1.4). □

(The same result holds in Banach spaces but requires a different proof: see Dugundji (1958, theorem 10.2). For the case where $\mathscr{X}^0 \neq \varnothing$, see the proof of 4.2.4. below.)

THEOREM 2.1.5 *If* $\mathscr{Y}$ *has the fixed point property and* $\mathscr{X}$ *is a retract of* $\mathscr{Y}$ *then* $\mathscr{X}$ *has the fixed point property.*

Proof. Let r be a retraction map of $\mathscr{Y}$ onto $\mathscr{X}$. If T is any continuous map of $\mathscr{X}$ into $\mathscr{X}$ then Tr is a continuous map of $\mathscr{Y}$ into $\mathscr{X}$. Since Tr maps $\mathscr{Y}$ into $\mathscr{Y}$, there is a fixed point w, thus $Trw = w$. Clearly $w \in \mathscr{X}$ so that $rw = w$ and hence $Tw = w$. □

DEFINITION 2.1.6 A topological space $\mathscr{X}$ is *contractible* (to a point x_0 in $\mathscr{X}$) if there exists a continuous function $f(x, t)$ on $\mathscr{X} \times [0, 1]$ to $\mathscr{X}$ such that $f(x, 0) \equiv x$ and $f(x, 1) \equiv x_0$.

In order to obtain a fairly intuitive proof of Brouwer's theorem we will assume known the following facts about homology groups. (In §2.2, we discuss some proofs of Brouwer's theorem which do not require homology theory; in Chapter 10 we refer to some short proofs requiring more algebraic topology.) We write S^n for the n-sphere and B^n for the closed n-ball.

REMINDER 2.1.7 With each complex $\mathscr{X}$ in Euclidean space and each integer $n \geqslant 1$, we can associate a unique group $H_n(\mathscr{X})$ (the *nth homology group* with integral coefficients). Also

(i) $H_n(S^n) = Z$, the group of integers;
(ii) if $\mathscr{X}$ is contractible, then $H_n(\mathscr{X}) = \{e\}$, the trivial group.