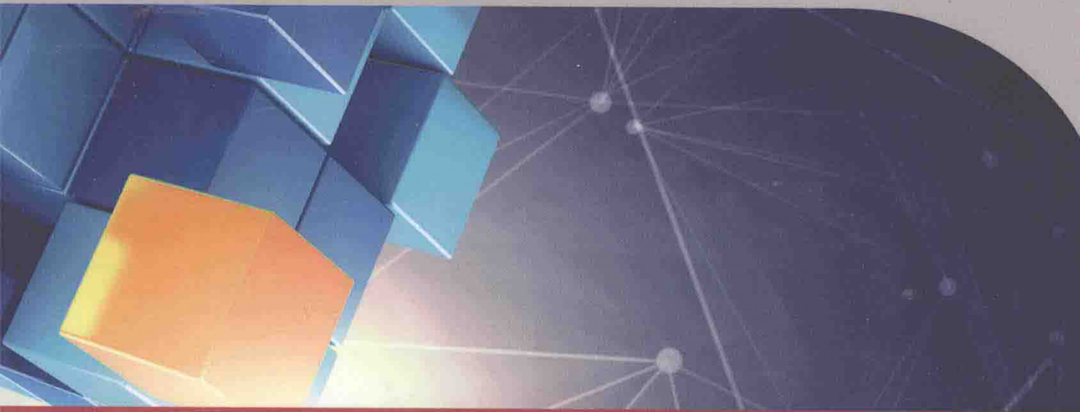
Control, Robotics and Sensors Series 107



Solved Problems in Dynamical Systems and Control

J. Tenreiro Machado, António M. Lopes, Duarte Valério and Alexandra M. Galhano

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The Institution of Engineering and Technology

Published by The Institution of Engineering and Technology, London, United Kingdom

The Institution of Engineering and Technology is registered as a Charity in England & Wales (no. 211014) and Scotland (no. SC038698).

First published 2016

The Institution of Engineering and Technology
Michael Faraday House
Six Hills Way, Stevenage
Herts, SG1 2AY, United Kingdom

www.theiet.org

British Library Cataloguing in Publication Data
A catalogue record for this product is available from the British Library

ISBN 978-1-78561-174-2 (hardback)
ISBN 978-1-78561-175-9 (PDF)

Typeset in India by MPS Limited
Printed in the UK by CPI Group (UK) Ltd, Croydon

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Chapter 1
Block diagram algebra and system transfer functions

1.1 Fundamentals

We introduce the Laplace transform as a method of converting differential equations in time into algebraic equations in a complex variable. Afterward, we present the concepts of transfer function and block diagram as a means to represent linear time-invariant (LTI) dynamical systems.

1.1.1 List of symbols

$b(t)$	feedback signal
$e(t)$	actuating error signal
$G(s)$, $H(s)$	transfer function
$\mathscr{L}$	Laplace operator
$r(t)$	time-domain input
s	Laplace variable
t	time
$y(t)$	time-domain output

1.1.2 Laplace transform and Laplace domain

If $f(t)$ is a piece-wise continuous function in time-domain, then its Laplace transform is given by [1]:

$$F(s) = \mathscr{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)dt \tag{1.1}$$

where $F(s)$ denotes a complex-valued function of the complex variable s and t represents time.

The Laplace transform converts linear time-domain differential equations into Laplace-domain (or complex s-domain) algebraic equations.

The inverse Laplace transform is:

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\sigma-jT}^{\sigma+jT} e^{st} F(s)ds \tag{1.2}$$

where $j = \sqrt{-1}$. The integration is done along the vertical line $Re(s) = \sigma$ in the complex plane, such that σ is greater than the real part of all singularities of $F(s)$. If $F(s)$ is a smooth function on $-\infty < Re(s) < \infty$, then σ can be set to zero.

Usually, given a rational function, $F(s)$, we adopt the partial fraction decomposition, or partial fraction expansion, method for expressing $F(s)$ as a sum of a polynomial and one, or several, fractions with simpler denominators. We then use the tables of Laplace transforms [2,3] to obtain $f(t)$ (see Appendix A).

1.1.3 Transfer function

A transfer function, $G(s)$, represents the dynamics of a LTI system [4]. Mathematically, it is the ratio between system output, $Y(s)$, and input, $R(s)$, in the Laplace domain, considering that all initial conditions and point equilibrium are zero: $G(s) = \frac{Y(s)}{R(s)}$.

The transfer function is an intrinsic property of a LTI system that represents the differential equation relating the system input to the output. Based on the transfer function, the system dynamic response can be determined for different inputs [2,5,6].

1.1.4 Block diagram

A block diagram is a graphic representation of a system dynamic model that includes no information about the physical construction of the system. Consequently, a block diagram representing a given system is not unique. Moreover, the main source of energy, as well as the energy flow in the system is not explicitly shown. Each individual block establishes a unilateral relationship between an input and an output signal, being assumed that there is no interaction between blocks.

Figure 1.1 depicts an example of a system block diagram. The ratio $G_{FF} = \frac{Y(s)}{E(s)} = G(s)$ between the output $Y(s)$ and the actuating error $E(s)$ is the feed-forward transfer function. The ratio $G_{OL} = \frac{B(s)}{E(s)} = G(s)H(s)$ between the feedback $B(s)$ and actuating error $E(s)$ signals represents the open-loop transfer function. Finally, the ratio between output $Y(s)$ and input $R(s)$ denotes the system closed-loop transfer function:

$$G_{CL} = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{1.3}$$

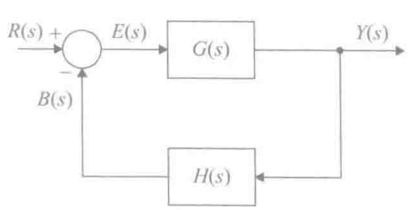

Figure 1.1 Example of a system block diagram

1.1.5 Block diagram algebra

Block diagrams can be systematically simplified by means of a general procedure that involves the identification of sub-diagrams, followed by block reduction according to given rules [3]. In Appendix A, we show several block diagrams that frequently occur in control systems, as well as the corresponding simplified blocks.

1.1.5.1 Mason rule

In complex block diagrams the transfer function, G, can be calculated by means of the Mason's rule [4]:

$$G = \frac{\sum_{k=1}^{N} G_k \Delta_k}{\Delta} \tag{1.4}$$

where

- $\Delta = 1 - \sum L_i + \sum L_i L_j - \sum L_i L_j L_k + \cdots + (-1)^m \sum \cdots + \cdots$.
- N: total number of forward paths between input and output.
- G_k: path gain of the kth forward path between input and output.
- L_i: loop gain of each closed loop in the system.
- $L_i L_j$: product of the loop gains of any two non-touching loops (no common nodes).
- $L_i L_j L_k$: product of the loop gains of any three pairwise non-touching loops.
- Δ_k: cofactor value of Δ for the kth forward path, with the loops touching the kth forward path removed.

1.2 Worked examples

Problem 1.1 Consider the block diagram of a control system in Figure 1.2. $R(s) = \mathscr{L}[r(t)]$ is the Laplace transform of the input and $Y(s) = \mathscr{L}[y(t)]$ is the Laplace transform of the output. The transfer function of the system $\frac{Y(s)}{R(s)}$ is:

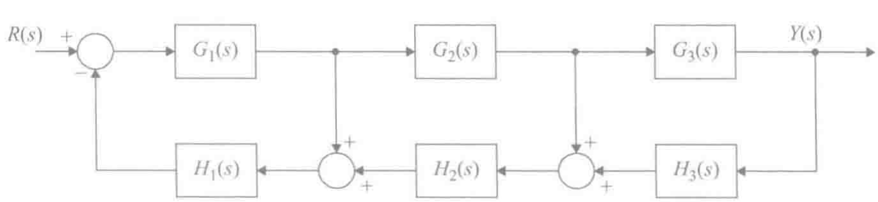

Figure 1.2 Block diagram of Problem 1.1

A) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1(s)G_2(s)G_3(s)}{1 + G_1(s)G_2(s)G_3(s)H_1(s)H_2(s)H_3(s)}$

B) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1(s)}{1 + G_1(s)H_1(s)} \cdot \dfrac{G_2(s)}{1 + G_2(s)H_2(s)} \cdot \dfrac{G_3(s)}{1 + G_3(s)H_3(s)}$

C) $\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s)}{1 + G_1(s)H_1(s)\{1 + G_2(s)H_2(s)[1 + G_3(s)H_3(s)]\}}$

D) None of the above.

Resolution Simplifying and combining blocks as in Figure 1.3 we get the system transfer function.

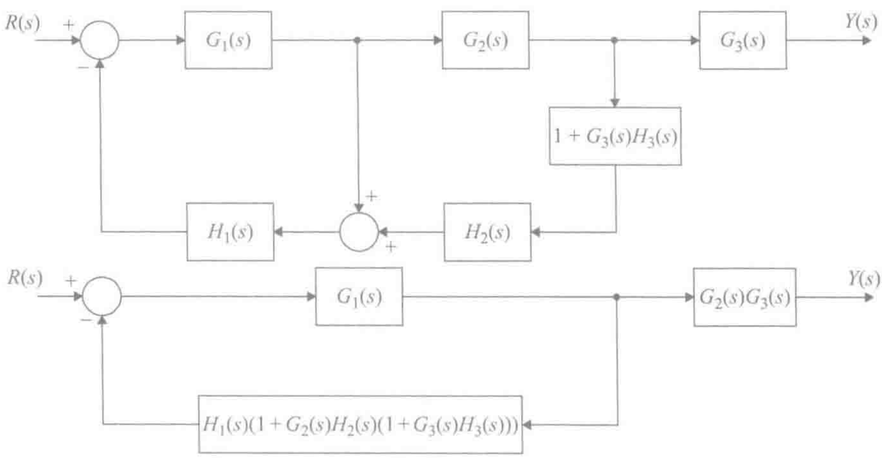

Figure 1.3 Resolution of Problem 1.1

Thus, the correct answer is option **C)**.

Problem 1.2 Consider the block diagram of a control system in Figure 1.4. The transfer function of the system $\frac{Y(s)}{R(s)}$ is:

A) $\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s) + H_2(s)}{1 + G_1(s)G_2(s)G_3(s)H_1(s)H_3(s)}$

B) $\frac{Y(s)}{R(s)} = \frac{G_1(s)[G_2(s)G_3(s) + H_2(s)]}{1 + G_1(s)G_2(s)G_3(s)H_1(s)H_3(s)}$

C) $\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s) + H_2(s)}{1 + G_1(s)H_1(s)[1 + G_2(s)G_3(s)H_3(s)]}$

D) None of the above.

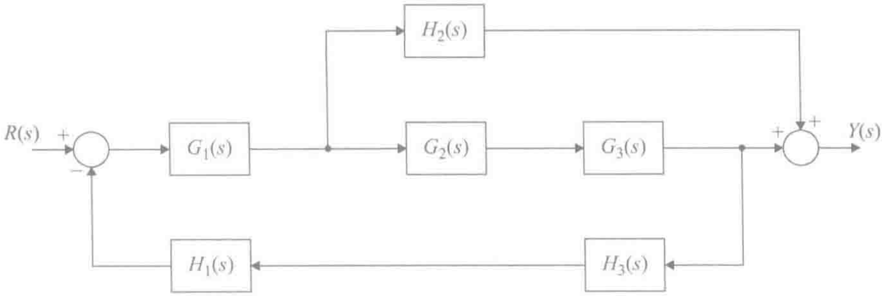

Figure 1.4 Block diagram of Problem 1.2

Resolution Simplifying and combining blocks as in Figure 1.5 we get the system transfer function.

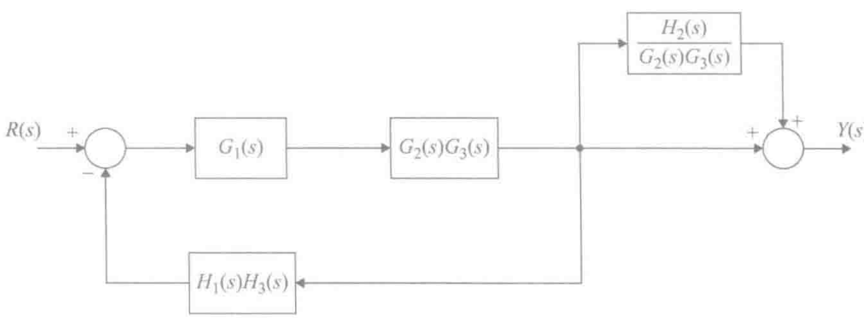

Figure 1.5 Resolution of Problem 1.2

Thus, the correct answer is option **B)**.

Problem 1.3 Consider the block diagram of a control system in Figure 1.6. The transfer function of the system $\dfrac{Y(s)}{R(s)}$ is:

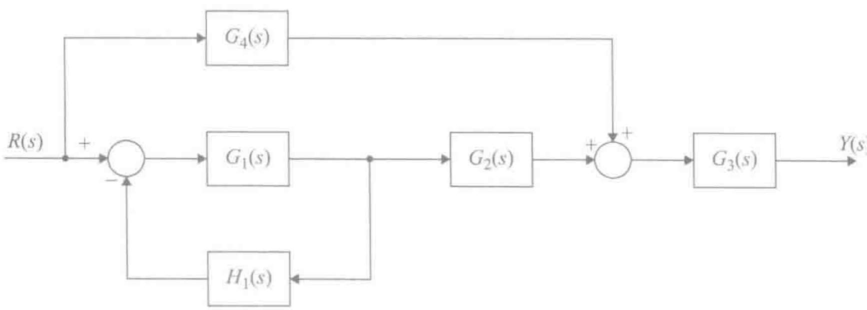

Figure 1.6 Block diagram of Problem 1.3

A) $$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)H_1(s)} + G_3(s)G_4(s)$$

B) $$\frac{Y(s)}{R(s)} = \left[\frac{G_1(s)G_2(s)}{1 + G_1(s)H_1(s)} + G_4(s)\right] G_3(s)$$

C) $$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)G_3(s)G_4(s)}{1 + G_1(s)H_1(s)}$$

D) $$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s) + G_3(s)G_4(s)}{1 + G_1(s)H_1(s)}.$$

Resolution Simplifying and combining blocks as in Figure 1.7 we get the system transfer function.

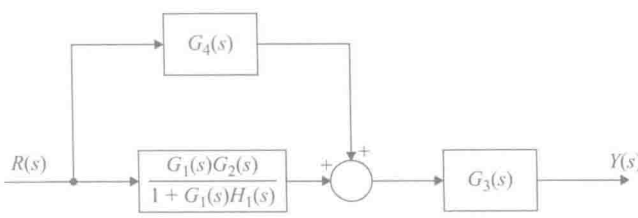

Figure 1.7 Resolution of Problem 1.3

Thus, the correct answer is option **B)**.

Problem 1.4 Consider the block diagram of a control system in Figure 1.8. The transfer function of the system $\frac{Y(s)}{R(s)}$ is:

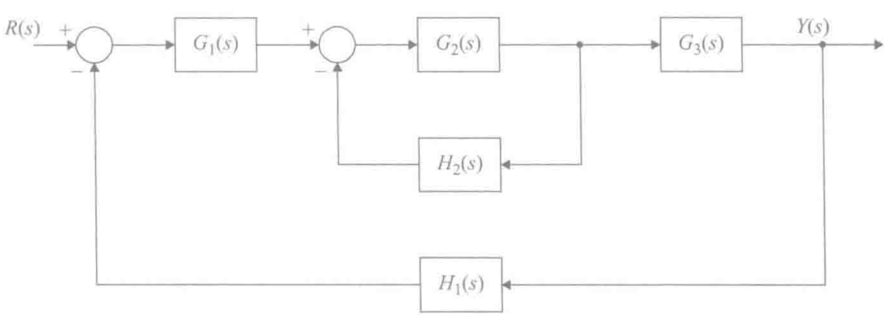

Figure 1.8 Block diagram of Problem 1.4

A) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1(s)G_2(s)G_3(s)}{1 + G_2(s)\left[H_2(s) + G_1(s)G_3(s)H_1(s)\right]}$

B) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1(s)G_2(s)G_3(s)}{1 + G_2(s)\left[H_2(s) + G_1(s)H_1(s)\right]}$

C) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1(s)G_2(s)G_3(s)}{1 + G_1(s)H_1(s) + G_2(s)H_2(s)}$

D) $\dfrac{Y(s)}{R(s)} = \dfrac{G_1(s)G_2(s)G_3(s)}{1 + G_2(s)H_2(s) + G_1(s)H_1(s)G_3(s)H_2(s)}$.

Resolution Simplifying and combining blocks as in Figure 1.9 we get the system transfer function.

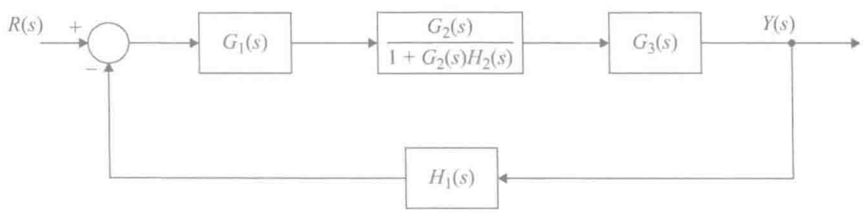

Figure 1.9 Resolution of Problem 1.4

Thus, the correct answer is option **A)**.

Problem 1.5 Consider the block diagram of a control system in Figure 1.10. Find the transfer function $\dfrac{Y(s)}{R(s)}$.

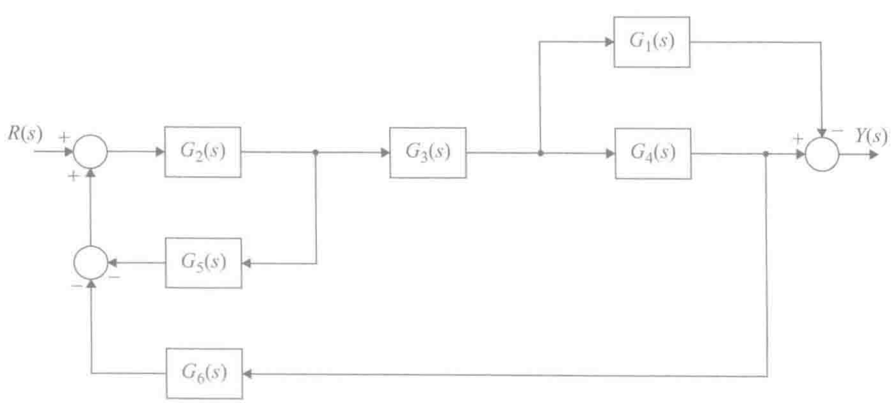

Figure 1.10 Block diagram of Problem 1.5

Resolution Simplifying and combining blocks as in Figure 1.11 we get the system transfer function.

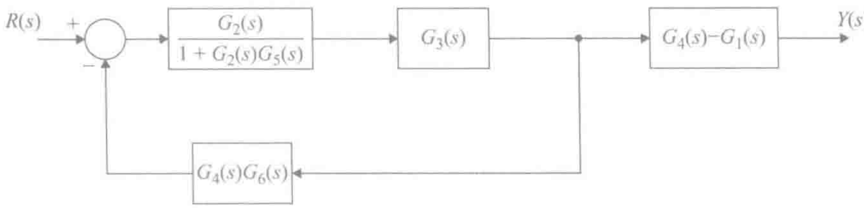

Figure 1.11 Resolution of Problem 1.5

Thus, $\frac{Y(s)}{R(s)} = \frac{G_2(s)G_3(s)\,[G_4(s) - G_1(s)]}{1 + G_2(s)G_5(s) + G_2(s)G_3(s)G_4(s)G_6(s)}$

Problem 1.6 Consider the block diagram of a system in Figure 1.12. The transfer function of the system $\frac{Y(s)}{R(s)}$ is:

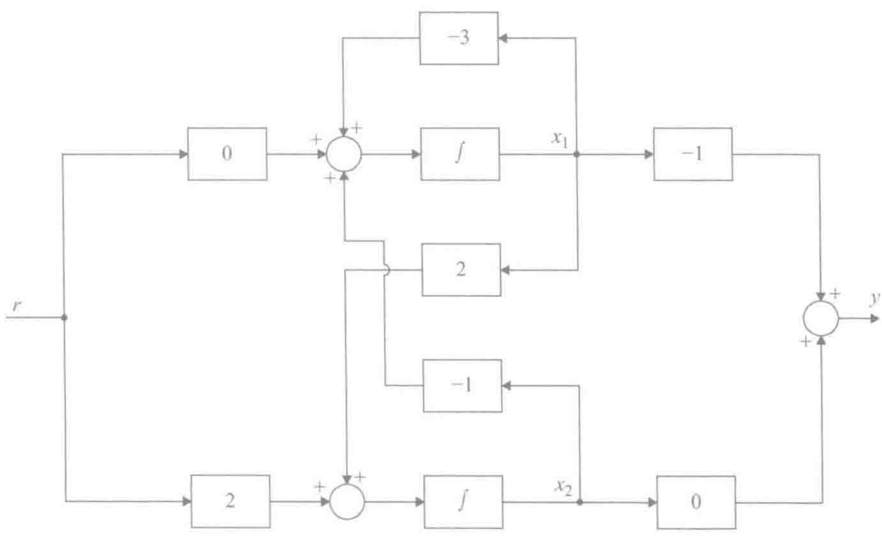

Figure 1.12 Block diagram of Problem 1.6

A) $\frac{Y(s)}{R(s)} = \frac{1}{(s+1)^2}$

B) $\frac{Y(s)}{R(s)} = \frac{2}{(s+1)(s+2)}$

C) $\dfrac{Y(s)}{R(s)} = \dfrac{3}{(s+1)(s+3)}$

D) None of the above.

Resolution Simplifying and combining blocks as in Figure 1.13 we get the system transfer function.

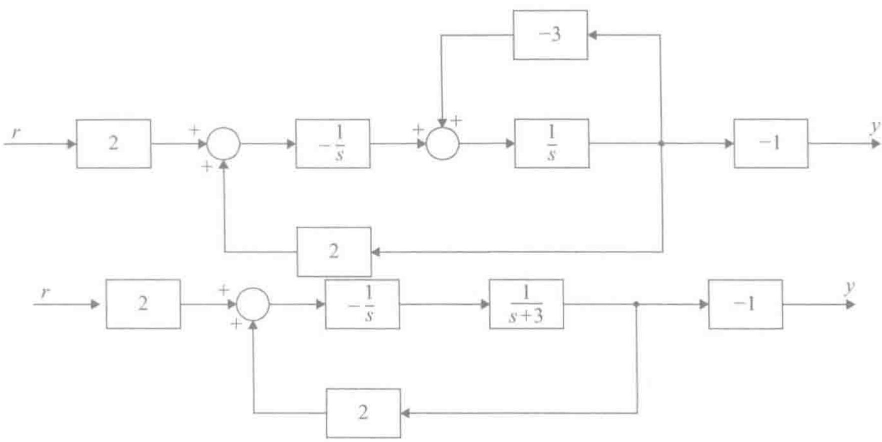

Figure 1.13 Resolution of Problem 1.6

Thus, the correct answer is option **B)**.

1.3 Proposed exercises

Exercise 1.1 Consider the block diagram in Figure 1.14.

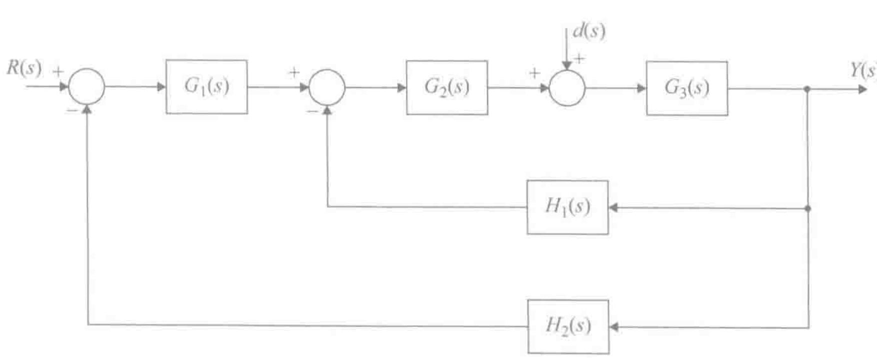

Figure 1.14 Block diagram of Exercise 1.1