Probability Distributions on Linear Spaces

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Editor's Preface to the English Edition

During the past twenty-five to thirty years the theory of probability in Banach and other linear topological spaces has developed very rapidly, and in a systematic fashion. One of the most important areas of research is concerned with the study of probability distributions (or measures) on Banach spaces. This English edition of Professor Vakhania's Russian text is the first systematic and clearly written introduction to the theory of probability distributions on linear spaces. This book can be used as a class or seminar text, for self-study, and as a reference work. It will be of interest to a broad audience of probabilists, functional analysts, measure theorists, statisticians, and a wide range of applied mathematicians who need results on probability distribution on linear spaces in connection with research on random equations and related topics.

Results presented in this book are extended in the following two papers by Professor Vakhania and his students: (1) S. A. Chobanyan and V. I. Tarieladez, "Gaussian characterization of certain Banach spaces." *J. Multivar. Anal.* 7: 183–203, 1977, and (2) N. N. Vakhania and V. I. Tarieladez, "Covariance operators of probability measures in locally convex spaces," *Theory Probab. Appl.* 23: 1–21, 1978. The reader is encouraged to study the above papers and to investigate the recent literature on the connections between probability theory in Banach spaces and their geometric properties.

Professor Vakhania has made some minor changes and corrections to this translation of the Russian edition by Professor I. I. Kotlarski. The series editor has made some changes in style.

A. T. Bharucha-Reid
Series Editor

Introduction

Probability distributions on linear spaces arise in probability theory in the following way. In the classical one-dimensional case, instead of investigating random variables, one can investigate probability distributions on the real line. According to this, in the infinite-dimensional case, for the process $\xi_t (t \in T)$, one should consider distributions on the family of all sample functions, that is, on the space R^T of all real functions on T. But it is only in the finite-dimensional case (when T is a finite set), that the analogy is sufficiently good. In the infinite-dimensional case problems arise that make little sense or are trivial in the finite-dimensional case. In reality, something known in advance about the random vector in R^n, does not necessarily help to simplify the sample space. For example, if it is known that the distribution is concentrated on some subspace, then we have again R^k, where $k < n$, but this is no simplification. In the case of a function space R^T, a priori information that the sample functions belong to some linear subspace can be used because different linear subspaces can have different natural topologies (which would not make sense for the whole R^T). This allows different additional possibilities of describing the process under consideration. Sometimes the same process can be embedded into different subspaces, thus varying the methods of investigation and the terms in which the process is described.

In this way, as a natural abstraction, there arises the notion of a probability distribution on an abstract linear space—in other words, the notion of a random element with values in a linear space.

There exists a different approach to these notions, which is disassociated from probability theory. This approach, which comes from the theory of functions and functional analysis, is motivated by the inner logic of developments in pure analysis. It is quite natural to extend the theory of integration to include functions defined on infinite-dimensional linear spaces, as well as functions of one and several variables. This endeavor immediately brings us to the notion of measures on linear spaces. Finite measures, normed by one, are called *probability measures*, or *probability distributions*.

In general, the problem of investigation of normed measures on linear spaces is very broad and has many different directions with or without emphasizing the possibility of probabilistic interpretation. Let us note the basic results of some of the directions, on which this book is based. Although all these investigations will not be covered, we shall go through a short description of the results that are given and therefore start with some background information.

In 1935 A. N. Kolmogorov [1] introduced the notion of a *characteristic functional* (i.e., of the Fourier transform) of the measure μ on the Banach space X (as an integral with respect to μ of $\exp[if(x)]$, $f \in X^*$). He gave the basic properties of the characteristic functional and pointed out the importance and possibility of further investigations in this area.

In 1951 M. Fréchet [2] examined Gaussian random elements (Gaussian distributions) in a Banach space. The random element $x \in X$ was called *Gaussian* if all real-valued random variables $f(x)$, $f \in X^*$, were Gaussian.

A systematic investigation of random elements with values in Banach spaces was begun in the works of E. Mourier and R. Fortet. In her basic work E. Mourier [3], in 1953, defined the expectation, using the weak integral (*Pettis integral*) of a random element. Further, for the case where X is a separable Hilbert space H, she defined an analogue of the variance, namely, the *covariance operator*, presupposing the existence of the expectation of the square of the norm of the random element. For Gaussian random elements in H, Mourier proved the existence of the expectation itself and of the expectation of the square of the norm, thus giving one of her basic results—the general form of the characteristic functionals of all Gaussian distributions on H.

The next steps in the investigation of distributions on Hilbert spaces are found in the works of Yu. V. Prohorov and V. V. Sazonov. In 1956 Yu. V. Prohorov [4] gave the analogue of Lévy's distance between distribution functions for the case of measures in metric spaces, and investigated the conditions for compactness and convergence of families of measures relative to this metric. Applying these results to distributions on Hilbert spaces, Prohorov obtained theorems expressing conditions for relative compactness of a family of distributions in terms of the corresponding family of characteristic functionals. These results served as the basis for a further result given by V. V. Sazonov [5] in 1958. He found that with respect to the topology on H, the analogue of Bochner's theorem is true: *A positive-definite normed complex functional on H is a characteristic functional of some distribution on H if and only if it is continuous with respect to this topology.*

These problems are closely related to the problem of extension of a weak distribution to a measure. While solving this problem, R. A. Minlos [6] independently obtained a theorem about the extension of generalized random processes, from which the analogue of Bochner's theorem follows for the distributions on spaces which are conjugate to countably normed Hilbertian nuclear spaces. The connection between the results of R. A. Minlos and V. V. Sazonov was perceived and analyzed by A. N. Kolmogorov [7].

Further references to other works connected to these problems will be given later, generally in the introductions to the chapters.

We will now briefly describe the book.

It contains four chapters. In the first chapter the general problems of probability distributions on the linear topological space R^N of all real numerical sequences with the usual linear operations and Tikhonov's topology are studied. After a preliminary description of the space R^N and its conjugate, we prove the analogues of the theorems of Bochner and Lévy for characteristic functionals.

The space R^N is first of all the natural sample space for random sequences. It is also interesting because a large class of linear spaces X may be embedded linearly and isomorphically into R^N, which allows reduction of problems from X to R^N. This process is particularly effective if X is a Hilbert space, because in this case its image is l_2, and the isomorphism is isometric (*Riesz–Fischer theorem*). Therefore all results proved for l_2 are also true for a general (separable) Hilbert space.

Further, we consider probability distributions on the Banach spaces $l_p (1 \leqslant p < \infty)$, c_0, and l_∞. We obtain some results analogous to Bochner's theorem under some conditions on the moments. Moreover, conditions of relative compactness of families of distributions on these spaces are investigated, along with the relations of these conditions to the properties of the corresponding families of characteristic functionals. Thus a result analogous to Lévy's theorem is obtained, which is the best possible in this situation. The basic idea in getting these results is the following: Each of the spaces l_p, c_0, l_∞, is considered as a linear subspace of R^N, and the theorem proved for R^N is applied. In this way the problem is reduced to obtaining conditions under which the distribution on R^N will be concentrated on a given subspace of R^N. The idea of enlarging the space is not, of course, new. However, we did not limit ourselves to considerations in the whole space. This space is a helping tool only and is not involved in the final formulations.

The method of embedding in R^N is also used in the second chapter, in which we investigate Gaussian distributions on R^N and its subspaces. First, the case of the whole space R^N is considered, and then we go to l_p, $1 \leqslant p < \infty$. After proving some auxiliary theorems we come to the basic theorem of this chapter: *In order that a Gaussian distribution* R^N *with* *expectation* $\{a_k\}$ *and covariance matrix* $\|s_{ij}\|$ *be concentrated on* l_p *(and hence to be Gaussian in* l_p*), it is necessary and sufficient that* $\{a_k\} \in l_p$ *and* $\{s_{kk}\} \in l_{p/2}$.

Then, we give two other formulations of this result. One is a theorem on the general form of characteristic functionals of all Gaussian distributions on l_p, which sheds new light on and is a generalization of the aforementioned result of E. Mourier. The other formulation gives an analogue of Bochner's theorem for Gaussian distributions on l_p by constructing a proper topology in the conjugate space. This topology is a natural generalization of Sazonov's topology.

Further, we show some other facts about Gaussian distributions on l_p, which are connected with different aspects of the theory of stochastic processes and their applications. By doing this we can consider these results as corollaries from the general theorems already proved, or as examples that illustrate their applications. We mention the proof of the exponential integrability of the square of the norm with respect to the Gaussian distribution, the central limit theorem that contains Mourier's theorem ($p = 2$) and Varadarajan's theorem ($p = 1$), the investigation of a stochastic differential equation of heat conduction with white noise on the right side (the Fourier transform of the solution of this equation determines the Gaussian distribution on R^N), and the characterization of nondegenerate Gaussian distributions (the nondegeneracy of μ is equivalent to the following property: The μ-measure of an arbitrary ball with a nonzero radius is positive).

Finally, in this chapter we investigate Gaussian distributions on the spaces c_0 and l_∞. We shall mention here only one result: For each $R > 0$, there exists on l_∞ a nondegenerate Gaussian distribution μ, such that the μ-measure of an arbitrary ball with radius R is zero.

The third chapter considers a separable Hilbert space H. Based on previous comments, the space H can be treated as l_2, hence we can use at once the results of previous theorems for this case. In this way we again obtain the known, as well as some new results.

Next, we come to some special problems, which are quite usual and traditional for the theory of probability. This part of the chapter is also connected with the methods and results of the second chapter. First, we consider the question of evaluating the rate of convergence in the central limit theorem. We consider the finite-dimensional case separately; and, in particular, when the coordinates of the vectors are independent, we obtain a uniform estimate in the class of all balls with the classical order for n (namely, $n^{-1/2}$), using the sum of Lyapunov's ratios as a parameter, which determines the dependence on the distribution. Then, we obtain in the case of a general Hilbert space, a uniform estimation of order $(\ln n)^{-1}$ on the class of ellipsoids with centers from a given ball, by operating on the distributions and applying the characteristic functionals.

Then, we consider the problem of evaluating large deviations for normed sums of independent random elements in H with bounded norms. An exponential type inequality is obtained for the evaluation of the measures of the complements of the ellipsoids that are defined by the nuclear operators.

Next in the third chapter we consider the distribution of the inner product of two Gaussian random elements in H. This problem is connected with the eigenvalue problem for a system of two operator equations. Particular cases are shown in which the solution is found in one form or another.

Finally, we define and give some simple properties of an integral with respect to a random measure with values in H, of a function whose values are linear operators $H \to H$. This integral is a random element in H; and we find, in particular, its covariance operator, which is in the form of a (nonrandom) integral in the strong sense (*Bochner's integral*).

The fourth chapter is devoted to some general problems of probability distributions on abstract Banach spaces. We discuss the notion of a characteristic functional and note its basic properties. We prove a theorem on conditions for the existence of the Pettis integral, which includes a theorem on the existence of the expectation of an arbitrary Gaussian distribution on a separable space X. Then, the *covariance operator* R is defined as a bounded linear transformation $X \to X^{**}$. The conditions for the existence of R are reduced to the natural necessary conditions; in particular, each Gaussian distribution has a covariance. The operators that transform a space into its conjugate have special properties, which distinguish them from the general transformations of one Banach space into another. For instance, the notions of symmetry and nonnegativeness can be applied to such operators, and all covariance operators have these properties. Under some broad assumptions the converse theorem is proved: *Every symmetric, nonnegative, bounded linear operator* $X^* \to X^{**}$ *is the covariance operator of some distribution on* X. The proof is based on the following factorization lemma, which is also of interest from other points of view: An arbitrary operator $R: X^* \to X^{**}$ with the preceding property can be factorized into the form $R = A^*A$, where A is a bounded linear transformation of the space X^* into some auxiliary Hilbert space (in some sense the operator A plays the role of the square root of R).

Next, the transformations given by the covariance operators are investigated; and, in particular, different sufficient conditions are considered under which $RX^* \subset X$ (in the sense of the natural embedding of X into X^{**}).

Further, we look at the problem of the characterization of covariance operators of Gaussian distributions. Using the factorization $R = A^*A$, we reduce the problem to that of characterizing bounded linear transformations $A: X^* \to H$, where H is an auxiliary Hilbert space. We define in two ways Hilbert–Schmidt linear transformations of a Banach space into a Hilbert space (and conversely, a Hilbert space into a Banach space). Correspondingly, we obtain two definitions for the operator $R : X^* \to X^{**}$ to be a *nuclear operator*, and we prove that the first condition of being a nuclear operator is necessary and the second is sufficient for R to belong to the class of Gaussian covariance operators. By making some strongly restrictive assumptions on X, we find that both definitions of a nuclear operator are identical, and hence we obtain the necessary and sufficient condition. The example of the space l_p ($p \neq 2$) shows immediately that the assumptions in the given definitions are not directly related to the existing general definition of a nuclear transformation of one Banach space into another.

Finally, we show that the conditions for an operator R to belong to the class of Gaussian covariance operators are of a topological nature: There exists a Hausdorff locally convex topology in X^*, which generalizes the topology induced in the second chapter for l_q, and the necessary and sufficient condition is the continuity of the quadratic functional $(Rf)(f)$ with respect to this topology.

Note that some of the results in the fourth chapter could be made more general by considering linear topological spaces instead of Banach spaces. However, the whole style of the book and its relatively elementary level would be lost by doing this, and it would be too high a price to pay for generalizations of some separate particular results.

I would like to thank Yu. V. Prohorov and V. V. Sazonov for our discussions on the results considered in this book, as well as on other problems. This has been an opportunity I have enjoyed for many years.

Contents

1

Characteristic Functionals of Probability Distributions on Spaces of Numerical Sequences

Introduction

Section 1.1 is a general introduction, which gives a description of the space R^N and its conjugate (dual). Note that not all the properties given here will be used later on. However, investigation of different types of infinite-dimensional spaces, in which the method of characteristic functionals is applied, should be considered not only for their particular interest but also as the possible steps along the way toward explaining the most general situations where these methods could be applied. From this point of view, it is desirable to underline those properties that help to apply the method of characteristic functionals in each such case. The importance of this method is connected with the theorems of Bochner and Lévy, which were first proved in the one-dimensional case.

In Section 1.2 it is shown that for a distribution on R^N both these theorems are true in the classical formulation. Then we go to Banach spaces $X = l_p$ $(1 \leqslant p < \infty)$, c_0, and l_∞. We consider X not as a measurable space itself but as a measurable subset of R^N, and we obtain conditions under which

a. The distribution on R^N will be concentrated on X.
b. A compact family of distributions on R^N, each of which is concentrated on X, will also be compact in X.

By this method, in Sections 1.3 and 1.4, we obtain different theorems of the type of Bochner and Lévy, which include as particular cases some of the results of Yu. V. Prohorov and V. V. Sazonov for Hilbert spaces (this is described more completely in Section 3.1).

The main results of the first chapter are published in references 8 and 9.

In connection with the results of this chapter we note the survey-type article by Yu. V. Prohorov [10], in which is given a quite adequate description (with a large bibliography) of results of the method of characteristic functionals of distributions on linear topological spaces.

1.1 The Space R^N: Basic Properties

1.1.1

Recall that R^N is defined as the linear space of all real numerical sequences with the natural group operation (addition) and multiplication by real numbers. We shall say that the element $0 \neq x \in R^N$ has length L, if $x_k = 0$ for $k > L$ and $x_L \neq 0$. If such an L does not exist, then the length of this element will be infinity. The length of the zero element will be zero.

We shall assign to the linear space R^N the following (Tikhonov's) topology, taking as a base of neighborhoods of the zero element the class of sets of the form

$$\Theta_{\epsilon,n}(0) = \bigcap_{k=1}^{n} \{x : |x_k| < \epsilon\}, \quad \epsilon > 0, \quad n \geqslant 1.$$

The continuity of linear operations may be verified easily. Thus we obtain a linear topological space, which will be denoted R^N. We note the basic properties of this space.

1. R^N has a countable base, and therefore it is separable.
2. Convergence in R^N is equivalent to coordinate convergence.
3. R^N is a Fréchet space (F-space), that is, it is locally convex, metrizable, and complete.

The proofs of (1) and (2) are simple. Local convexity is obvious. To show that the space is metrizable we can take the following function as the metric.

$$\rho(x, y) = \sum_{k=1}^{\infty} \alpha_k \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

where the α_k are positive numbers that form a convergent series. The completeness is obvious.

4. R^N is additionally a *Montel–Fréchet space*, that is, along with (3), each closed bounded[1] set in R^N is compact.[2]

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[1] The set $M \subset R^N$ being *bounded* is taken to mean: For each neighborhood of zero $\Theta_{\epsilon,n}(0)$ there exists $\lambda > 0$ such that

$$\{\lambda x : x \in M\} \subset \Theta_{\epsilon,n}(0).$$

[2] Note that because of the countable base, compactness and sequential compactness are equivalent for sets in R^N.

PROOF. The conclusion is a consequence of the well-known Bolzano–Weierstrass theorem and the fact that compactness and boundedness of sets in R^N mean correspondingly coordinatewise compactness and coordinatewise boundedness.

Note here that for relative compactness (also for boundedness) of a set M in R^N it follows (see also reference 11, p. 346) that it is necessary and sufficient that

$$\text{for all } x \in M, \qquad |x_k| \leqslant A_k, A_k > 0 \quad (k = 1, 2, \ldots).$$

From this it follows that the space R^N is not locally compact. □

Now we want to find a space conjugate to R^N. By a *conjugate space* we mean a linear space (without fixing a topology) of all continuous linear functionals defined on R^N, using the usual operations of addition and scalar multiplication by reals (real numbers). Denote by R_0^N the subset of the space R^N, which consists of all elements with finite lengths.

5. The conjugate to R^N is R_0^N.

PROOF. It is sufficient to show that an arbitrary continuous linear functional f on R^N has the form $f(x) = \sum_k f_k x_k$, where the f_k are the coordinates of a fixed element $f \in R_0^N$ and are uniquely determined by the functional f. Obviously, x can be represented in the form

$$x = \sum_{k=1}^{n} x_k e^{(k)} + r^{(n)}x,$$

where $e^{(k)} = \{e_i^{(k)}\}$, with $e_i^{(k)} = 0$ for $i \neq k$ and $e_k^{(k)} = 1$, and the first n coordinates of the element $r^{(n)}x$ are equal to zero. Hence $r^{(n)}x \to 0$ as $n \to \infty$ for each element $x \in R^N$. Now, using the additivity and continuity of the functional f, and denoting the real number $f(e^{(k)})$ by f_k, we obtain from the previous equality the representation $f(x) = \sum_{k=1}^{\infty} f_k x_k$. We need to show that the element $f = \{f_k\}$ has a finite length. If this is not the case, then there exists an infinite subsequence $e^{(k_i)}$ such that $|f_{k_i}| > 0$. Then, $g^{(k_i)} = e^{(k_i)}/f_{k_i} \to 0$ for $i \to \infty$, whereas $f(g^{(k_i)}) = 1$ for all i, which contradicts the continuity. □

1.1.2

In the linear space R_0^N, which is a conjugate to a Fréchet space, one can introduce ([12], p. 74, although in this case it can be done explicitly) a locally convex topology I_c (called the *topology of uniform convergence on compact sets*) by defining as a base of neighborhoods of zero the family of sets of the form

$$U_{\epsilon,K}(0)=\left\{f:\sup_{x\in K}|f(x)|<\epsilon\right\},$$

where ϵ is an arbitrary positive number and K is an arbitrary compact set in R^N.

1. The topology I_c can also be obtained from another base of neighborhoods of zero by taking sets

$$V_{\{\epsilon_k\}}(0)=\sum_{k=1}^{\infty}\{f:|f_k|<\epsilon_k\},$$

where $\{\epsilon_k\}$ is an arbitrary sequence of positive numbers that converges to zero.

As a matter of fact, it is easy to see that $V_{\{\epsilon_k\}}(0)\subset U_{\epsilon,K}(0)$, if, for instance, $\epsilon_k=(\epsilon/k^2)A_k'$, where $A_k'=\max(1,A_k)$ and A_k $(k=1,2,\ldots)$ are numbers determined by the compact set K (see property (4) in Subsection 1.1.1. Conversely, $U_{\epsilon,K}(0)\subset V_{\{\epsilon_k\}}(0)$, if $\epsilon/A_k<\epsilon_k$ and K is the product of closed intervals $[0,A_k]$, because it is clear that in this case

$$\sup_{x\in K}|f(x)|\geqslant\sup_k|f_k|A_k.$$

2. The convergence of a sequence of elements $f^{(n)}\in R_0^N$ in the I_c-topology is equivalent to satisfying the following two conditions.
   a. Coordinatewise convergence.
   b. Boundedness of the lengths of the elements $f^{(n)}$.

   Suppose $f^{(n)}\to 0$, because it is surely sufficient to consider only convergence to the zero element. The coordinatewise convergence is obvious. If the sequence of the lengths is not bounded, then there exists infinite subsequences of indices $\{n_j\}$ and $\{k_j\}$ such that $|f_{k_j}^{(n_j)}|>0$. But then none of the elements $f^{(n_j)}$ will be in the neighborhood of zero $V_{\{\epsilon_j\}}(0)$ if $\epsilon_j\leqslant|f_{k_j}^{(n_j)}|$, $j=1,2,\ldots$, which contradicts the assumption that $f^{(n)}\to 0$. The converse is similarly easy to prove.
3. This convergence reduces to convergence in the so-called *weakest topology*, that is, convergence of $\{f^{(n)}\}$ as the weak convergence of linear functionals. The fact that convergence of $\{f^{(n)}(x)\}$ for all $x\in R^N$ follows from the convergence of $f^{(n)}$ in the I_c-topology is a simple consequence of being able to take the limit under the finite sum. The converse is nontrivial; that is, weak convergence implies convergence in the I_c-topology. For simplicity we consider only convergence to zero. The coordinatewise convergence follows from the convergence along the elements $e^{(k)}$. We shall show that the lengths are bounded. Without loss of generality, we assume the opposite—that the sequence of lengths $\{L_k\}$ corresponding to $\{f^{(k)}\}$ is strictly increasing. Then, we have $|f_{L_k}^{(k)}|>0$, $k=1,2,\ldots$ and $L_k=+\infty$. But $f^{(k)}(x)\to 0$ for all x.

We now construct an element $a \in R^N$ on which the indicated sequence of functionals will not converge to zero, resulting in a contradiction. We shall determine the element a by means of the following relations (their solvability is obvious).

$$a_k = 0 \quad \text{for} \quad k \neq L_j \quad (j = 1, 2, \ldots)$$
$$a_{L_1} f_{L_1}^{(1)} = 1,$$
$$a_{L_1} f_{L_1}^{(2)} + a_{L_2} f_{L_2}^{(2)} = 1$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots$$
$$a_{L_1} f_{L_1}^{(n)} + a_{L_2} f_{L_2}^{(n)} + \cdots + a_{L_n} f_{L_n}^{(n)} = 1,$$

It is easy to see that $f^{(n)}(a) = 1$ for each n, and thus $f^{(n)}$ does not converge to zero along this element.

4. The convergence in the I_c-topology implies coordinatewise convergence as previously mentioned. The converse is not true (the simplest example is: $f_k^{(n)} = 1/n$ for $k \leqslant n$, $f_k^{(n)} = 0$ for $k > n$). This shows that the I_c-topology in R_0^N is stronger than the Tikhonov topology (induced from R^N).
5. The linear space R_0^N with the I_c-topology is a linear topological space. This space as a strong conjugate[3] to a Montel space is also a Montel space ([12], p. 90). The fact that in R_0^N boundedness is equivalent to compactness can be verified explicitly, noting that each compact (bounded) set in R_0^N consists of elements that have their lengths bounded by a common L and that the first L coordinates determine a compact (and respectively bounded) set in the Euclidean L-dimensional space. From this it follows, in particular, that the linear topological space (R_0^N, I_c) is not locally compact.
6. The conjugate to R_0^N is R^N. The topology of uniform convergence on I_c-compact sets in R_0^N (i.e., the topology $I_c = I_b$ in R^N) is identical with the original topology.

The proof is very simple, hence we omit it.

1.1.3

We now need to introduce the class of measurable sets in the space R^N. Sets in R^N will be called *measurable* if they belong to the minimal σ-algebra which contains all *Borel cylinders*, that is, sets of the form

$$\{x : x \in R^N, (x_1, x_2, \ldots, x_n) \in B\},$$

[3] By *strong conjugate* we mean the conjugate space with the *topology I_b of uniform convergence on bounded sets*. But it is obvious that $I_b = I_c$ in the case when the original space is a Montel space.