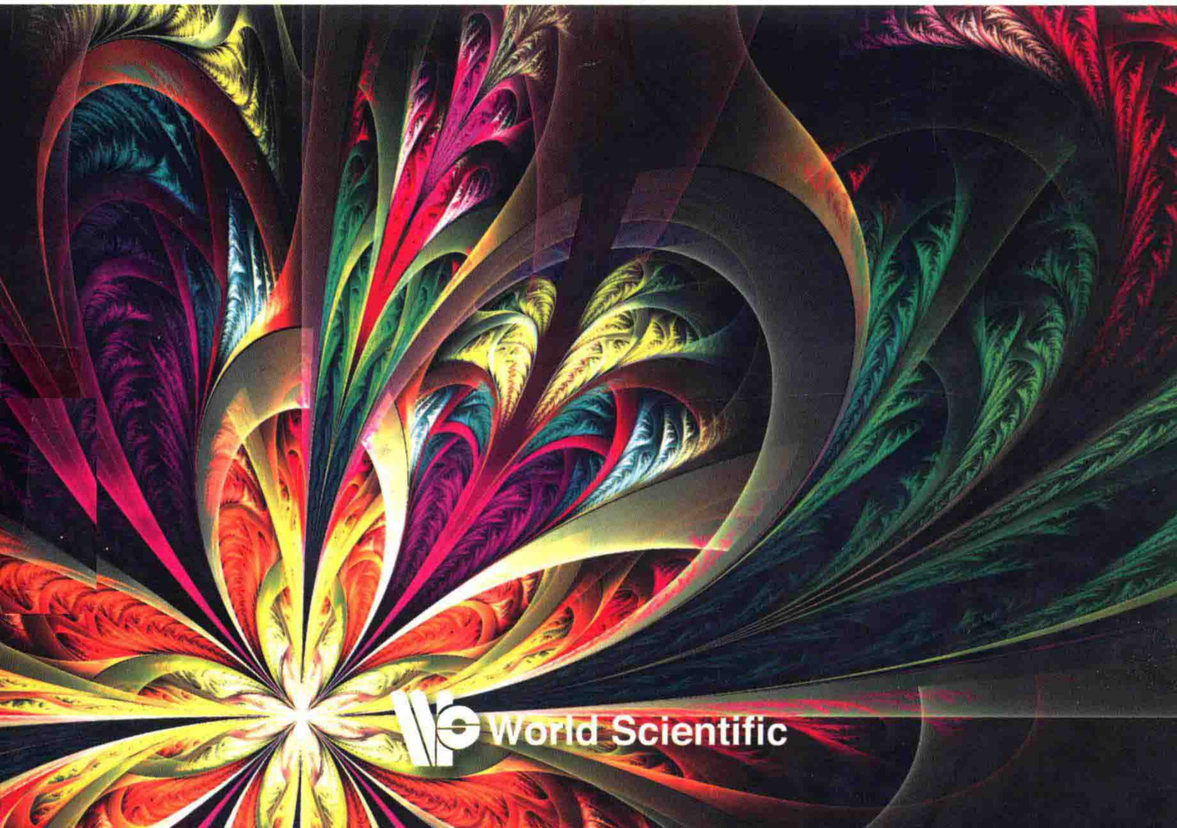
Series on Concrete and Applicable Mathematics – Vol. 16

George A. Anastassiou

Frontiers in Approximation Theory



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Frontiers in Approximation Theory

George A. Anastassiou

University of Memphis, USA

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Frontiers in Approximation Theory

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Series Editor: Professor George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, USA

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"The Main feature of life on this Planet is continuation. That is continuous movement of everything which involves stops, gaps, and jumps."

Preface

In this monograph we present recent work of last five years of the author in Approximation Theory. It is the natural outgrowth of his related publications. Chapters are self-contained and advanced courses can be taught out of this book. An extensive list of references is given per chapter.

The topics covered are diverse. The first eight chapters are dedicated to fractional monotone approximation theory introduced for the first time by the author, taking the related ordinary theory of usual differentiation at the fractional differentiation level having polynomials and splines as approximators. Very little is written so far about fractional approximation theory which is at its infancy. Chapters 9–10 are dedicated to the approximation by discrete singular operators of Favard style, e.g. of Picard and Gauss–Weierstrass types. We continue with Chapter 11 which is about the approximation by interpolating operators induced by neural networks, a connection with computer science, a very detailed and extensive work covering all aspects of the topic. We finish with Chapter 12 about approximation theory and functional analysis on time scales, a very modern topic, detailing all the pros and cons of the approach.

The book's results are expected to find applications in many areas of pure and applied mathematics. As such this monograph is suitable for researchers, graduate students, and seminars of the above subjects, also to be in all science libraries.

The preparation of book took place during 2014–2015 in Memphis, TN, USA.

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George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
USA
March 1, 2015

Contents

Chapter 1

Fractional Monotone Approximation

Let $f \in C^p([-1,1])$, $p \geq 0$ and let L be a linear left fractional differential operator such that $L(f) \geq 0$ throughout $[0,1]$. We can find a sequence of polynomials Q_n of degree $\leq n$ such that $L(Q_n) \geq 0$ over $[0,1]$, furthermore f is approximated uniformly by Q_n. The degree of this restricted approximations is given by an inequalities using the modulus of continuity of $f^{(p)}$.

This chapter follows [3].

1.1 Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1.1. *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1(f^{(p)}, x)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1.1}$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{1.2}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1] \tag{1.3}$$

and

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)|\leq Cn^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{1.4}$$

where C is independent of n or f.

We need

Theorem (of Trigub see [7; 8]) Let $n\in\mathbb{N}$. Be given a real function g, with $g^{(p)}$ continuous in $[-1,1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that $\max_{-1\leq x\leq 1}\left|g^{(j)}(x)-q_n^{(j)}(x)\right|\leq R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right)$, $j=0,1,...,p$, where R_p is independent of n or g.

In this chapter we extend Theorem 1.1 to the fractional level. Now L is a linear left Caputo fractional differential operator. Here the monotonicity property is only true on the critical interval $[0,1]$. Quantitative uniform approximation remains true on all of $[-1,1]$.

To the best of our knowledge this is the first time fractional monotone Approximation Theory is introduced.

We need and make

Definition 1.1. ([4], p. 50) Let $\alpha>0$ and $\lceil\alpha\rceil=m$, ($\lceil\cdot\rceil$ ceiling of the number). Consider $f\in C^m([-1,1])$. We define the left Caputo fractional derivative of f of order α as follows:

$$\left(D_{*-1}^{\alpha}f\right)(x)=\frac{1}{\Gamma(m-\alpha)}\int_{-1}^{x}(x-t)^{m-\alpha-1}f^{(m)}(t)\,dt, \tag{1.5}$$

for any $x\in[-1,1]$, where Γ is the gamma function.

We set

$$D_{*-1}^{0}f(x)=f(x),$$
$$D_{*-1}^{m}f(x)=f^{(m)}(x),\ \forall x\in[-1,1]. \tag{1.6}$$

1.2 Main Result

We present

Theorem 1.2. *Let h,k,p be integers, $0\leq h\leq k\leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)},\delta\right)$, $\delta>0$, there. Let $\alpha_j(x)$, $j=h,h+1,...,k$ be real functions, defined and bounded on $[-1,1]$ and assume for $x\in[0,1]$ that $\alpha_h(x)$ is either $\geq$ some number $\alpha>0$ or $\leq$ some number $\beta<0$. Let the real numbers $\alpha_0=0<\alpha_1\leq 1<\alpha_2\leq 2<...<\alpha_p\leq p$. Here $D_{*-1}^{\alpha_j}f$ stands for the left Caputo fractional derivative of f of order α_j anchored at -1. Consider the linear left fractional differential operator*

$$L:=\sum_{j=h}^{k}\alpha_j(x)\left[D_{*-1}^{\alpha_j}\right] \tag{1.7}$$

and suppose, throughout $[0,1]$,

$$L(f) \geq 0. \tag{1.8}$$

Then, for any $n \in \mathbb{N}$, *there exists a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \quad \textit{throughout} \quad [0,1], \tag{1.9}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{1.10}$$

where C *is independent of* n *or* f.

Proof. Let $n \in \mathbb{N}$. By the theorem of Trigub given a real function g, with $g^{(p)}$ continuous in $[-1,1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} \left| g^{(j)}(x) - q_n^{(j)}(x) \right| \leq R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right), \tag{1.11}$$

$j = 0, 1, ..., p$, where R_p is independent of n or g.

Here $h, k, p \in \mathbb{Z}_+$, $0 \leq h \leq k \leq p$.

Let $\alpha_j > 0$, $j = 1, ..., p$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 ... < ... < \alpha_p \leq p$. That is $\lceil \alpha_j \rceil = j$, $j = 1, ..., p$.

We consider the left Caputo fractional derivatives

$$\left(D_{*-1}^{\alpha_j} g \right)(x) = \frac{1}{\Gamma(j - \alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} g^{(j)}(t)\, dt, \tag{1.12}$$

$$\left(D_{*-1}^{j} g \right)(x) = g^{(j)}(x),$$

and

$$\left(D_{*-1}^{\alpha_j} q_n \right)(x) = \frac{1}{\Gamma(j - \alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} q_n^{(j)}(t)\, dt, \tag{1.13}$$

$$\left(D_{*-1}^{j} q_n \right)(x) = q_n^{(j)}(x); \ j = 1, ..., p,$$

where Γ is the gamma function

$$\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt, \quad v > 0. \tag{1.14}$$

We notice that

$$\begin{aligned} &\left| \left(D_{*-1}^{\alpha_j} g \right)(x) - \left(D_{*-1}^{\alpha_j} q_n \right)(x) \right| \\ &= \frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} g^{(j)}(t)\, dt - \int_{-1}^{x} (x-t)^{j-\alpha_j-1} q_n^{(j)}(t)\, dt \right| \\ &= \frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left(g^{(j)}(t) - q_n^{(j)}(t) \right) dt \right| \end{aligned} \tag{1.15}$$

$$\leq \frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left| g^{(j)}(t) - q_n^{(j)}(t) \right| dt \tag{1.16}$$

$$\overset{(1.11)}{\leq} \frac{1}{\Gamma(j-\alpha_j)} \left(\int_{-1}^{x} (x-t)^{j-\alpha_j-1} dt \right) R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right)$$

$$= \frac{1}{\Gamma(j-\alpha_j)} \frac{(x+1)^{j-\alpha_j}}{(j-\alpha_j)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right) \tag{1.17}$$

$$\leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right).$$

We proved that for any $x \in [-1, 1]$ we have

$$\left| \left(D_{*-1}^{\alpha_j} g \right)(x) - \left(D_{*-1}^{\alpha_j} q_n \right)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right). \tag{1.18}$$

Hence it holds

$$\max_{-1 \leq x \leq 1} \left| \left(D_{*-1}^{\alpha_j} g \right)(x) - \left(D_{*-1}^{\alpha_j} q_n \right)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right), \tag{1.19}$$

$j = 0, 1, ..., p$.

Above we set $D_{*-1}^{0} g(x) = g(x)$, $D_{*-1}^{0} q_n(x) = q_n(x)$, $\forall\, x \in [-1, 1]$, and $\alpha_0 = 0$, i.e. $\lceil \alpha_0 \rceil = 0$.

Put

$$s_j \equiv \sup_{-1 \leq x \leq 1} \left| \alpha_h^{-1}(x) \alpha_j(x) \right|, \quad j = h, ..., k, \tag{1.20}$$

and

$$\eta_n := R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \left(\sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-p} \right). \tag{1.21}$$

I. Suppose, throughout $[0, 1]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1 \leq x \leq 1} \left| D_{*-1}^{\alpha_j} \left(f(x) + \eta_n (h!)^{-1} x^h \right) - \left(D_{*-1}^{\alpha_j} Q_n \right)(x) \right|$$

$$\overset{(1.19)}{\leq} \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p n^{j-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{1.22}$$

$j = 0, 1, ..., p$.

In particular ($j = 0$) holds

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) + \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{1.23}$$

and

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|\leq \eta_n (h!)^{-1}+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right)$$

$$=(h!)^{-1}R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)\left(\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}n^{j-p}\right)$$

$$+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right) \tag{1.24}$$

$$\leq R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)n^{k-p}\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right). \tag{1.25}$$

That is

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|$$

$$\leq R_p\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right)n^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{1.26}$$

proving (1.10).

Here

$$L=\sum_{j=h}^{k}\alpha_j(x)\left[D_{*-1}^{\alpha_j}\right],$$

and suppose, throughout $[0,1]$, $Lf\geq 0$.

So over $0\leq x\leq 1$, using (1.12) to compute $D_{*-1}^{\alpha_j}x^h$, (1.21) for η_n and (1.22), we get

$$\alpha_h^{-1}(x)L(Q_n(x))=\alpha_h^{-1}(x)L(f(x))+\eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \tag{1.27}$$

$$+\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{*-1}^{\alpha_j}Q_n(x)-D_{*-1}^{\alpha_j}f(x)-\frac{\eta_n}{h!}D_{*-1}^{\alpha_j}x^h\right]$$

$$\geq \eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\left(\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}n^{j-p}\right)R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)$$

$$=\eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\eta_n=\eta_n\left[\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-1\right] \tag{1.28}$$

$$= \eta_n \left[\frac{(x+1)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq \eta_n \left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq 0. \quad (1.29)$$

Explanation: We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h - \alpha_h < 1$ and $1 \leq h - \alpha_h + 1 < 2$. Thus $\Gamma(h - \alpha_h + 1) \leq 1$ and

$$1 - \Gamma(h - \alpha_h + 1) \geq 0. \quad (1.30)$$

Hence

$$L(Q_n(x)) \geq 0, x \in [0, 1]. \quad (1.31)$$

II. Suppose, throughout $[0, 1]$, $\alpha_h(x) \leq \beta < 0$. In this case let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} \left| D_{*-1}^{\alpha_j} \left(f(x) - \eta_n (h!)^{-1} x^h \right) - \left(D_{*-1}^{\alpha_j} Q_n \right)(x) \right| \quad (1.32)$$

$$\leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p n^{j-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right),$$

$j = 0, 1, ..., p$.

In particular holds $(j = 0)$

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) - \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (1.33)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \eta_n (h!)^{-1} + R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right)$$

$$\overset{\text{(as before)}}{\leq} R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) n^{k-p} \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right). \quad (1.34)$$

That is

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)|$$

$$\leq R_p \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right) n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (1.35)$$

reproving (1.10).

Again suppose, throughout $[0, 1]$, $Lf \geq 0$.

Also if $0 \leq x \leq 1$, then

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) - \eta_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \quad (1.36)$$

$$+ \sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[D_{*-1}^{\alpha_j} Q_n(x) - D_{*-1}^{\alpha_j} f(x) + \frac{\eta_n}{h!} \left(D_{*-1}^{\alpha_j} x^h \right) \right]$$