AN ELEMENTARY TREATISE

ON THE

DYNAMICS OF A PARTICLE AND OF RIGID BODIES

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PREFACE

IN the following work I have tried to write an elementary class-book on those parts of Dynamics of a Particle and Rigid Dynamics which are usually read by Students attending a course of lectures in Applied Mathematics for a Science or Engineering Degree, and by Junior Students for Mathematical Honours. Within the limits with which it professes to deal, I hope it will be found to be fairly complete.

I assume that the Student has previously read some such course as is included in my Elementary Dynamics. I also assume that he possesses a fair working knowledge of Differential and Integral Calculus; the Differential Equations, with which he will meet, are solved in the Text, and in an Appendix he will find a summary of the methods of solution of such equations.

In Rigid Dynamics I have chiefly confined myself to two-dimensional motion, and I have omitted all reference to moving axes.

I have included in the book a large number of Examples, mostly collected from University and College Examination Papers; I have verified every question, and hope that there will not be found a large number of serious errors.

For any corrections, or suggestions for improvement, I shall be grateful.

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Solutions of the Examples have now been published.

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DYNAMICS OF A PARTICLE

DYNAMICS OF A RIGID BODY

CHAPTER I

FUNDAMENTAL DEFINITIONS AND PRINCIPLES

1. The velocity of a point is the rate of its displacement, so that, if P be its position at time t and Q that at time $t + \Delta t$, the limiting value of the quantity $\frac{PQ}{\Delta t}$, as Δt is made very small, is its velocity.

Since a displacement has both magnitude and direction, the velocity possesses both also; the latter can therefore be represented in magnitude and direction by a straight line, and is hence called a vector quantity.

2. A point may have two velocities in different directions at the same instant; they may be compounded into one velocity by the following theorem known as the Parallelogram of Velocities;

If a moving point possess simultaneously velocities which are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, they are equivalent to a velocity which is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.

Thus two component velocities AB, AC are equivalent to the resultant velocity AD, where AD is the diagonal of the parallelogram of which AB, AC are adjacent sides.

If BAC be a right angle and $BAD = \theta$, then $AB = AD \cos\theta$, $AC = AD \sin\theta$, and a velocity v along AD is equivalent to the two component velocities $v\cos\theta$ along AB and $v\sin\theta$ along AC.

Triangle of Velocities. If a point possess two velocities completely represented (*i.e.* represented in magnitude, direction and sense) by two straight lines AB and BC, their resultant is completely represented by AC. For completing the parallelogram $ABCD$, the velocities AB, BC are equivalent to AB, AD whose resultant is AC.

Parallelopiped of Velocities. If a point possess three velocities completely represented by three straight lines OA, OB, OC their resultant is, by successive applications of the parallelogram of velocities, completely represented by OD, the diagonal of the parallelopiped of which OA, OB, OC are conterminous edges.

Similarly OA, OB and OC are the component velocities of OD.

If OA, OB, and OC are mutually at right angles and u, v, w are the velocities of the moving point along these directions, the resultant velocity is $\sqrt{u^2+v^2+w^2}$ along a line whose direction cosines are proportional to u, v, w and are thus equal to

$$\frac{u}{\sqrt{u^2+v^2+w^2}}, \quad \frac{v}{\sqrt{u^2+v^2+w^2}} \text{ and } \frac{w}{\sqrt{u^2+v^2+w^2}}.$$

Similarly, if OD be a straight line whose direction cosines referred to three mutually perpendicular lines OA, OB, OC are l, m, n, then a velocity V along OD is equivalent to component velocities lV, mV, nV along OA, OB, and OC respectively.

3. Change of Velocity. Acceleration. If at any instant the velocity of a moving point be represented by OA, and at any subsequent instant by OB, and if the parallelogram $OABC$ be completed whose diagonal is OB, then OC or AB represents the velocity which must be compounded with OA to give OB, *i.e.* it is the change in the velocity of the moving point.

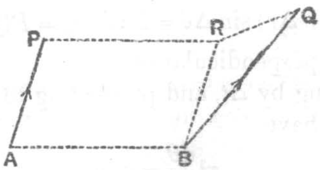
Acceleration is the rate of change of velocity, *i.e.* if OA, OB represent the velocities at times t and $t+\Delta t$, then the limiting value of $\dfrac{AB}{\Delta t}$ (*i.e.* the limiting value of the ratio of the change in the velocity to the change in the time), as Δt becomes indefinitely small, is the acceleration of the moving point. As in the case of velocities, a moving point may possess simultaneously accelerations in different directions, and they may be compounded into one by a theorem known as the Parallelogram of Accelerations similar to the Parallelogram of Velocities.

As also in the case of velocities an acceleration may be resolved into two component accelerations.

The results of Art. 2 are also true for accelerations as well as velocities.

4. Relative Velocity. When the distance between two moving points is altering, either in direction or in magnitude or in both, each point is said to have a velocity relative to the other.

Suppose the velocities of two moving points A and B to be represented by the two lines AP and BQ (which are not necessarily in the same plane), so that in the unit of time the positions of the points would change from A and B to P and Q. Draw BR equal and parallel to AP. The velocity BQ is, by the Triangle of Velocities, equivalent to the velocities BR, RQ, *i.e.* the velocity of B is equivalent to the velocity of A together with a velocity RQ.



The velocity of B relative to A is thus represented by RQ.

Now the velocity RQ is equivalent to velocities RB and BQ (by the Triangle of Velocities), *i.e.* to velocities completely represented by BQ and PA.

Hence *the velocity of B relative to A is obtained by compounding the absolute velocity of B with a velocity equal and opposite to that of A.*

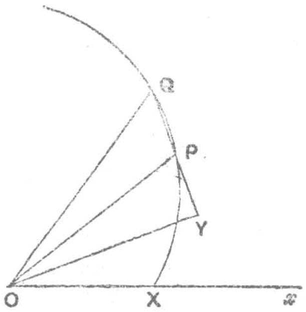
Conversely, since the velocity BQ is equivalent to the velocities BR and RQ, *i.e.* to the velocity of A together with the velocity of B relative to A, therefore *the velocity of any point B is obtained by compounding together its velocity relative to any other point A and the velocity of A.*

The same results are true for accelerations, since they also are vector quantities and therefore follow the parallelogram law.

5. Angular velocity of a point whose motion is in one plane.

If a point P be in motion in a plane, and if O be a fixed point and Ox a fixed line in the plane, the rate of increase of the angle xOP per unit of time is called the angular velocity of P about O.

Hence, if at time t the angle xOP be θ, the angular velocity about O is $\frac{d\theta}{dt}$.



If Q be the position of the point P at time $t + \Delta t$, where Δt is small, and v the velocity of the point at time t, then

$$v = \text{Lt.} \frac{PQ}{\Delta t}.$$

If

$$\angle POQ = \Delta\theta, \text{ and } OP = r,\ OQ = r + \Delta r,$$

then

$$r(r + \Delta r)\sin\Delta\theta = 2\Delta POQ = PQ \,.\, OY,$$

where OY is the perpendicular on PQ.

Hence, dividing by Δt, and proceeding to the limit when Δt is very small, we have

$$r^2 \frac{d\theta}{dt} = v \,.\, p,$$

where p is the perpendicular from O upon the tangent at P to the path of the moving point.

Hence, if ω be the angular velocity, we have $r^2\omega = v \,.\, p$.

The angular acceleration is the rate at which the angular velocity increases per unit of time, and

$$= \frac{d}{dt}(\omega) = \frac{d}{dt}\left(\frac{v \,.\, p}{r^2}\right).$$

Areal velocity. The areal velocity is, similarly, the rate at which the area XOP increases per unit of time, where X is the point in which the path of P meets Ox. It

$$= \text{Lt.} \frac{\text{area } POQ}{\Delta t} = \tfrac{1}{2} r^2 \,.\, \omega.$$

6. Mass and Force. Matter has been defined to be "that which can be perceived by the senses" or "that which can be acted upon by, or can exert, force." It is like time and space a primary conception, and hence it is practically impossible to give it a precise definition. A body is a portion of matter bounded by surfaces.

A particle is a portion of matter which is indefinitely small in all its dimensions. It is the physical correlative of a geometrical point. A body which is incapable of any rotation, or which moves without any rotation, may for the purposes of Dynamics, be often treated as a particle.

The mass of a body is the quantity of matter it contains.

A force is that which changes, or tends to change, the state of rest, or uniform motion, of a body.

7. If to the same mass we apply forces in succession, and they generate the same velocity in the same time, the forces are said to be equal.

If the same force be applied to two different masses, and if it produce in them the same velocity in the same time, the masses are said to be equal.

It is here assumed that it is possible to create forces of equal intensity on different occasions, *e.g.* that the force necessary to keep a given spiral spring stretched through the same distance is always the same when other conditions are unaltered.

Hence by applying the same force in succession we can obtain a number of masses each equal to a standard unit of mass.

8. Practically, different units of mass are used under different conditions and in different countries.

The British unit of mass is called the Imperial Pound, and consists of a lump of platinum deposited in the Exchequer Office.

The French, or Scientific, unit of mass is called a gramme, and is the one-thousandth part of a certain quantity of platinum, called a Kilogramme, which is deposited in the Archives.

$$\text{One gramme} = \text{about } 15\cdot432 \text{ grains}$$
$$= \cdot0022046 \text{ lb.}$$
$$\text{One Pound} = 453\cdot59 \text{ grammes.}$$

9. The units of length employed are, in general, either a foot or a centimetre.

A centimetre is the one-hundredth part of a metre which
= 39·37 inches
= 3·2809 ft. approximately.

The unit of time is a second. 86400 seconds are equal to a mean solar day, which is the mean or average time taken by the Earth to revolve once on its axis with regard to the Sun.

The system of units in which a centimetre, gramme, and second are respectively the units of length, mass, and time is called the C.G.S. system of units.

10. The density of a body, when uniform, is the mass of a unit volume of the body, so that, if m is the mass of a volume V of a body whose density is ρ, then $m = V\rho$. When the density is variable, its value at any point of the body is equal to the limiting value of the ratio of the mass of a very small portion of the body surrounding the point to the volume of that portion, so that

$$\rho = \text{Lt.}\frac{m}{V}, \text{ when } V \text{ is taken to be indefinitely small.}$$

The weight of a body at any place is the force with which the earth attracts the body. The body is assumed to be of such finite size, compared with the Earth, that the weights of its component parts may be assumed to be in parallel directions.

If m be the mass and v the velocity of a particle, its Momentum is mv and its Kinetic Energy is $\frac{1}{2}mv^2$. The former is a vector quantity depending on the direction of the velocity. The latter does not depend on this direction and such a quantity is called a Scalar quantity.

11. Newton's Laws of Motion.

Law I. Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it be compelled by impressed force to change that state.

Law II. The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts.

Law III. Tc every action there is an equal and opposite reaction.

These laws were first formally enunciated by Newton in his *Principia* which was published in the year 1686.

12. If P be the force which in a particle of mass m produces an acceleration f, then Law II states that

$$P = \lambda \frac{d}{dt}(mv), \text{ where } \lambda \text{ is some constant,}$$

$$= \lambda m f.$$

If the unit of force be so chosen that it shall in unit mass produce unit acceleration, this becomes $P = mf$.

If the mass be not constant we must have, instead,

$$P = \frac{d}{dt}(mu).$$

The unit of force, for the Foot-Pound-Second system, is called a Poundal, and that for the C.G.S. system is called a Dyne.

13. The acceleration of a freely falling body at the Earth's surface is denoted by g, which has slightly different values at different points. In feet-second units the value of g varies from 32·09 to 32·25 and in the C.G.S. system from 978·10 to 983·11. For the latitude of London these values are 32·2 and 981 very approximately, and in numerical calculations these are the values generally assumed.

If W be the weight of a mass of one pound, the previous article gives that

$$W = 1 \, . \, g \text{ poundals,}$$

so that the weight of a lb. = 32·2 poundals approximately.

So the weight of a gramme = 981 dynes nearly.

A poundal and a dyne are absolute units, since their values are the same everywhere.

14. Since, by the Second Law, the change of motion produced by a force is in the direction in which the force acts, we find the combined effect of a set of forces on the motion of a particle by finding the effect of each force just as if the other forces did not exist, and then compounding these effects. This is the principle known as that of the *Physical Independence of Forces.*

From this principle, combined with the Parallelogram of Accelerations, we can easily deduce the Parallelogram of Forces.

15. Impulse of a force. Suppose that at time t the value of a force, whose direction is constant, is P. Then the impulse of the force in time τ is defined be $\int_0^\tau P\,.\,dt$.

From Art. 12 it follows that the impulse

$$= \int_0^\tau m\frac{dv}{dt}\,dt = \Big[mv\Big]_0^\tau$$

= the momentum generated in the direction of the force in time τ.

Sometimes, as in the case of blows and impacts, we have to deal with forces, which are very great and act for a very short time, and we cannot measure the magnitude of the forces. We measure the effect of such forces by the momentum each produces, or by its impulse during the time of its action.

16. Work. The work done by a force is equal to the product of the force and the distance through which the point of application is moved in the direction of the force, or, by what is the same thing, the product of the element of distance described by the point of application and the resolved part of the force in the direction of this element. It therefore $=\int Pds$, where ds is the element of the path of the point of application of the force during the description of which the force in the direction of ds was P.

If X, Y, Z be the components of the force parallel to the axes when its point of application is (x, y, z), so that $X = P\frac{dx}{ds}$, etc. then

$$\int(X\,dx + Y\,dy + Z\,dz) = \int(P\frac{dx}{ds}\,dx + \ldots + \ldots)$$

$$= \int P\left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right\} ds = \int P\,ds$$

= the work done by the force P.

The theoretical units of work are a **Foot-Poundal** and an **Erg**. The former is the work done by a poundal during a displacement of one foot in the direction of its action; the latter is that done by a Dyne during a similar displacement of one cm.

One Foot-Poundal = 421390 Ergs nearly. One Foot-Pound is the work done in raising one pound vertically through one foot.

17. **Power.** The rate of work, or Power, of an agent is the work that would be done by it in a unit of time.

The unit of power used by Engineers is a Horse-Power. An agent is said to be working with one Horse-Power, or H.P., when it would raise 33,000 pounds through one foot per minute.

18. The Potential Energy of a body due to a given system of forces is the work the system can do on the body as it passes from its present configuration to some standard configuration, usually called the zero position.

For example, since the attraction of the Earth (considered as a uniform sphere of radius a and density ρ) is known to be $\gamma \cdot \frac{4\pi a^3 \rho}{3} \cdot \frac{1}{x^2}$ at a distance x from the centre, the potential energy of a unit particle at a distance y from the centre of the Earth, $(y > a)$,

$$= \int_y^a \left(- \frac{4\pi\gamma\rho a^3}{3x^2} \right) dx = \frac{4}{3} \pi\gamma\rho a^3 \left(\frac{1}{a} - \frac{1}{y} \right),$$

the surface of the earth being taken as the zero position.

19. From the definitions of the following physical quantities in terms of the units of mass, length, and time, it is clear that their dimensions are as stated.

Quantity	Mass	Dimensions in length	Time
Volume Density	1	−3	
Velocity		1	−1
Acceleration		1	−2
Force	1	1	−2
Momentum	1	1	−1
Impulse	1	1	−1
Kinetic Energy	1	2	−2
Power or Rate of Work	1	2	−3
Angular Velocity			−1

CHAPTER II

MOTION IN A STRAIGHT LINE

20. Let the distance of a moving point P from a fixed point O be x at any time t. Let its distance similarly at time $t + \Delta t$ be $x + \Delta x$, so that $PQ = \Delta x$.

The velocity of P at time t

$$= \text{Limit, when } \Delta t = 0, \text{ of } \frac{PQ}{\Delta t}$$

$$= \text{Limit, when } \Delta t = 0, \text{ of } \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

Hence the velocity $v = \dfrac{dx}{dt}$.

Let the velocity of the moving point at time $t + \Delta t$ be

$$v + \Delta v.$$

Then the acceleration of P at time t

$$= \text{limit, when } \Delta t = 0, \text{ of } \frac{\Delta v}{\Delta t}$$

$$= \frac{dv}{dt}$$

$$= \frac{d^2x}{dt^2}.$$

21. Motion in a straight line with constant acceleration f.

Let x be the distance of the moving point at time t from a fixed point in the straight line.

Then $$\frac{d^2x}{dt^2} = f \quad \text{..........................(1).}$$

Hence, on integration, $$v = \frac{dx}{dt} = ft + A \quad \text{..................(2),}$$

where A is an arbitrary constant.

Integrating again, we have

$$x = \tfrac{1}{2}ft^2 + At + B \quad \text{....................(3),}$$

where B is an arbitrary constant.

Again, on multiplying (1) by $2\frac{dx}{dt}$, and integrating with respect to t, we have

$$v^2 = \left(\frac{dx}{dt}\right)^2 = 2fx + C \quad \text{..................(4),}$$

where C is an arbitrary constant.

These three equations contain the solution of all questions on motion in a straight line with constant acceleration. The arbitrary constants A, B, C are determined from the initial conditions.

Suppose for example that the particle started at a distance a from a fixed point O on the straight line with velocity u in a direction away from O, and suppose that the time t is reckoned from the instant of projection.

We then have that when $t = 0$, then $v = u$ and $x = a$. Hence the equations (2), (3), and (4) give

$$u = A, \quad a = B, \text{ and } u^2 = C + 2fa.$$

Hence we have $$v = u + ft,$$

$$x - a = ut + \tfrac{1}{2}ft^2,$$

and $$v^2 = u^2 + 2f(x - a),$$

the three standard equations of Elementary Dynamics.

22. *A particle moves in a straight line OA starting from rest at A and moving with an acceleration which is always directed towards O and varies as the distance from O; to find the motion.*

Let x be the distance OP of the particle from O at any time t; and let the acceleration at this distance be μx.

The equation of motion is then

$$\frac{d^2x}{dt^2} = -\mu x \quad \text{.......................(1).}$$

[We have a negative sign on the right-hand side because $\frac{d^2x}{dt^2}$ is the acceleration in the direction of x increasing, *i.e.* in the direction OP; whilst μx is the acceleration towards O, *i.e.* in the direction PO.]

Multiplying by $2\frac{dx}{dt}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C.$$

If OA be a, then $\frac{dx}{dt} = 0$ when $x = a$, so that $0 = -\mu a^2 + C$,

and $$\therefore \left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2).$$

$$\therefore \frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2 - x^2} \quad \text{..................(2).}$$

[The negative sign is put on the right-hand side because the velocity is clearly negative so long as OP is positive and P is moving towards O.]

Hence, on integration,

$$t\sqrt{\mu} = -\int \frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\frac{x}{a} + C_1,$$

where $$0 = \cos^{-1}\frac{a}{a} + C_1, \quad i.e. \ C_1 = 0,$$

if the time be measured from the instant when the particle was at A.

$$\therefore x = a\cos\sqrt{\mu}t \quad \text{....................(3).}$$

When the particle arrives at O, x is zero; and then, by (2), the velocity $= -a\sqrt{\mu}$. The particle thus passes through O and immediately the acceleration alters its direction and tends to diminish the velocity; also the velocity is destroyed on the left-hand side of O as rapidly as it was produced on the right-hand side; hence the particle comes to rest at a point A' such that OA and OA' are equal. It then retraces its path, passes through O, and again is instantaneously at rest at A. The whole